Spectral Statistics in Nondiffusive Regimes

Alexander Altland* and Yuval Gefen
Department of Nuclear Physics, The Weizmann Institute of Science, 76100 Rehovot, Israel
(Received 19 November 1992)

We present an efficient approach for calculating spectral correlations of finite disordered systems beyond the diffusion approximation. For both ballistic (but not perfectly clean) and diffusive systems we find new regimes of spectral correlations. We address the nonuniversal features of these correlations and interpret them in terms of semiclassical concepts.

PACS numbers: 71.25.Mg, 03.65.Sq, 05.45.+b

A major theme in the study of mesoscopic systems has been the manifestation of quantum interference phenomena on submicron scales in the form of novel thermodynamic effects. Significant progress has been made once connections with level statistics and spectral correlations in such systems have been pointed out [1–9]. Subsequently, the close links between the theory of quantum chaos and certain aspects of mesoscopic physics have been elucidated [10–12].

Some important observable quantities may be related to the level statistics of isolated mesoscopic systems with random potentials. Examples include the ac conductivity $\sigma(\omega)$, expressed in terms of level correlations over an energy scale ω [11], the typical (single sample) persistent current, which is closely connected with level correlations over the Thouless energy E_c [12] and the canonically averaged magnetization of quantum dots subject to an external magnetic field, which is related to the fluctuations of the level number over a field dependent energy scale [5,7,10]. Furthermore, one may hope that some information concerning the level structure of small finite conductors would become available by means of direct measurements (e.g., ac absorption or tunneling experiments).

A particularly useful quantity characterizing level spectra is the two-level correlation function

$$Y_2(E_1,H_1;E_2,H_2)=\frac{1}{\nu^2}[\langle\nu(E_1,H_1)\nu(E_2,H_2)\rangle - \langle\nu(E_1,H_1)\rangle\langle\nu(E_2,H_2)\rangle], \quad (1)$$

where $\nu(E,H)$ is the sample specific density of states at energy E and applied field H, the angular brackets denote averaging over the random potential and $\nu\equiv\langle\nu(E_f)\rangle$. A variety of thermodynamic quantities can be derived from this correlation function, including the root-mean-square sample-to-sample fluctuations in the number of single particle levels within an energy interval ΔE (near the Fermi energy) at an applied field H, $\Sigma_2(\Delta E,H)$.

Besides its importance for practical applications in condensed matter physics, the function Y_2 also plays a major role as an interface between the theory of disordered systems on the one hand and the semiclassical theory of quantum chaos on the other hand. According to Ref. [10], the energy Fourier transform of Y_2 in $\omega\equiv E_1-E_2$ satisfies, $Y_2(t)\propto tP(t)$, where $P(t)$ denotes the classical probability for periodic motion. In this interpretation, Y_2 contains information about the scattering of electrons moving along periodic orbits of length v_f/ω, v_f being the Fermi velocity.

The functions Y_2 and Σ_2 have so far been calculated only for a limited range of parameters. Denoting the system's linear size by L, the elastic mean free path by l, the Fermi energy (momentum) by E_f (p_f), the diffusion constant by D, and the system's dimensionality by d, one defines several important energy scales: the average interlevel spacing Δ, $E_c=D/L^2$, the inverse elastic mean free time $1/\tau=v_f/l$, and E_f. A diffusive system is defined by $\Delta\ll E_c\ll 1/\tau$ [or equivalently $1\ll L/l\ll(p_fL)^{d-1}$]. For this range of parameters the behavior of $\Sigma_2(\Delta E,0)$ is known for the ergodic regime, $\Delta\ll\Delta E\ll E_c$, where random matrix theory applies [1], and for the diffusive [3] regime, $E_c\ll\Delta E\ll 1/\tau$. The effect of weak and intermediate fields on Σ_2 has also been studied [3,5,7,9]. However, a comprehensive description of thermodynamics on mesoscopic scales requires an extension of this picture to include large energy intervals, wider ranges of disorder, and stronger magnetic fields.

Here we report on a new approach to the calculation of the two level correlation function (and, more generally, of other thermodynamic quantities and fluctuations thereof). Although our approach is perturbative, it goes beyond the scope of the standard diffusion approximation which employs an effective diffusion equation to describe fluctuations in both the particle-hole and the Cooper channel. Presenting first an outline of this approach, we apply it then to the study of the zero field Σ_2. Specifically, (1) we apply our method to the large energy ($\Delta E\gg 1/\tau$) regime, finding a new type of behavior which differs substantially from both the random matrix and the diffusive regimes, and (2) for the first time we calculate Σ_2 for systems which are ballistic ($L<l$ or equivalently $1/\tau<v_F/L$) but already nonperturbative in the sense that the disorder is strong enough to mix neighboring levels and cannot be treated by means of elementary perturbation theory [$l<L(p_fL)^{d-1}$ or $\Delta<1/\tau$]. The relevance of the latter condition has first been discussed by Sivan and Imry in Ref. [13]. Again we find new regimes of lev-

0031-9007/93/71(20)/3339(4)$06.00
© 1993 The American Physical Society

el correlations.

We now sketch our calculation of the two-level correlation function and the level number fluctuations

$$\Sigma_2(\Delta E;H_1,H_2)=\frac{1}{\Delta^2}\int_{E-\Delta E/2}^{E+\Delta E/2}dE_1dE_2Y_2(E_1,H_1;E_2,H_2),\quad E=\mathcal{O}(E_f)\,, \tag{2}$$

for a system of noninteracting electrons moving in a disorder potential V [14]. To begin with, we model the disorder by a white noise potential $\langle V(x)V(y)\rangle=(2\pi\nu\tau)^{-1}\delta(x-y)$. The generalization to potentials with finite correlation length will be discussed below. Expressed in terms of advanced and retarded single electron Green functions $\mathcal{G}^{\pm}(E,H)=(E^{\pm}-\mathcal{H})^{-1}$, where $\mathcal{H}$ is the full Hamiltonian of the system and $E^{\pm}=E\pm i\gamma$, Y_2 reads

$$Y_2(E_1,H_1;E_2,H_2)=\frac{1}{2\pi^2\nu^2L^{2d}}\Re\langle\mathrm{tr}\mathcal{G}^{+}(E_1,H_1)\,\mathrm{tr}\mathcal{G}^{-}(E_2,H_2)\rangle_c\,, \tag{3}$$

where the subscript c indicates the omission of all disconnected contributions to the average. In order to evaluate Eq. (3) by means of diagrammatic perturbation theory we represent $\mathcal{G}$ as $\mathcal{G}^{\pm}(E,H)=\partial_E\ln(E^{\pm}-\mathcal{H})$, and take the inelastic broadening $\gamma\approx\Delta$ thereby avoiding divergences at $\omega\to 0$. Expanding $\mathcal{G}$ in terms of V and averaging over disorder we obtain to leading order in p_fl,

$$Y_2(E_1,H_1;E_2,H_2)=\frac{1}{2\pi^2\nu^2L^{2d}}\partial^2_{E_1,E_2}\Re\left[\sum_{n=2}^{\infty}\frac{1}{n}[S_n^{(D)}(E_1-E_2,H_-)+S_n^{(C)}(E_1-E_2,H_+)]+S_1(E_1-E_2)\right], \tag{4}$$

where $H_{\pm}=H_1\pm H_2$. A diagrammatic representation of the S_n is shown in Fig. 1. Introducing the quantities

$$\begin{aligned}\zeta^{(D)}(r,r')&=\frac{1}{2\pi\nu\tau L^d}G^{+}(E_1,H_1;r,r')G^{-}(E_2,H_2;r',r)\,,\\ \zeta^{(C)}(r,r')&=\frac{1}{2\pi\nu\tau L^d}G^{+}(E_1,H_1;r,r')G^{-}(E_2,H_2;r,r')\,,\end{aligned} \tag{5}$$

where G denotes the disorder averaged Green functions, and regarding them as coordinate representation of linear operators $\zeta^{(C,D)}$, we may write $S_n^{(C,D)}=\mathrm{tr}[\zeta^{(C,D)}]^n$. Consequently

$$Y_2(\omega,H_1,H_2)=\frac{1}{2\pi^2\nu^2L^{2d}}\partial^2_{\omega}(\mathrm{tr}\{\hat{\ln}[1-\zeta^{(D)}(\omega,H_-)]+\hat{\ln}[1-\zeta^{(C)}(\omega,H_+)]\}-S_1(\omega))\,, \tag{6}$$

where $\hat{\ln}(1-x)\equiv\ln(1-x)+x$ and the notation indicates that the weak dependence of Y_2 on the center coordinate $(E_1+E_2)/2$ is inessential for our purposes. According to Eq. (6) the analysis of the spectral correlation function amounts to analyzing the eigenvalues of the integral operator ζ. This is the main advantage of representing Y_2 in terms of energy derivatives of S_n. Expanding the Green functions entering Eq. (3) directly rather than their logarithm, leads to diagrams containing vertex parts in addition to impurity ladders (cf. Ref. [3]) which are more difficult to evaluate in the nondiffusive regimes. Moreover, the energy integrals in Eq. (2) over energy ranges larger than $1/\tau$ then become virtually intractable.

The eigenvalues of ζ depend on the sample's dimensionality and shape. We consider a quadratic two dimensional system, with Dirichlet boundary conditions on the Green functions (no current flow across the boundaries). Although our approach may account for magnetic fields [14], we shall consider here the field free case $H_1=H_2=0$. Because of the boundary conditions imposed on our model system, the diagonalization of the operators ζ turns out to be intricate and will be discussed elsewhere [14]. As a result we find the eigenvalues (cf. also Ref. [15]) $\lambda_q(\omega)=[(1+i\omega\tau)^2+(ql)^2]^{-1/2}$ where $\mathbf{q}=\pi L^{-1}(m_1,m_2)$, $m_{1,2}=0,1,2,\ldots$ and $q=|\mathbf{q}|$. In the field free case no distinction between $\zeta^{(C)}$ and $\zeta^{(D)}$ is necessary. We get

$$\Sigma_2(\Delta E)=\frac{1}{\pi^2}\Re\sum_{\mathbf{q}}\left[2\hat{\ln}\left(\frac{1-\lambda_q(\Delta E)}{1-\lambda_q(0)}\right)-[S_1(\Delta E)-S_1(0)]\right] \tag{7}$$

from which Y_2 may be obtained according to $Y_2(\omega)$

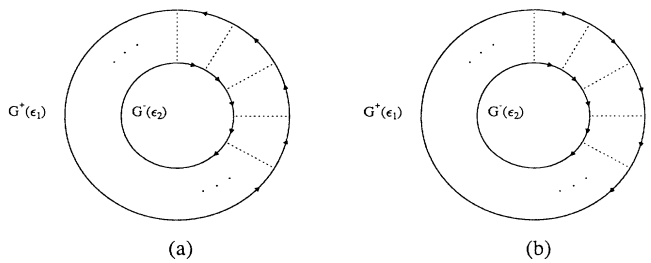

FIG. 1. Diagrammatic representation of $S_n^{(D)}$ (a) and $S_n^{(C)}$ (b). The inner and outer rings are connected by, respectively, n impurity lines. The massive arrows represent impurity averaged Green functions. For S_1 and S_2 the direction of the arrows is meaningless and one should avoid double counting.

$=(\Delta^2/2)\partial^2_{\Delta E}\Sigma_2(\Delta E)|_{\Delta E=\omega}$. In the following we evaluate these expressions for diffusive and ballistic systems separately.

Diffusive systems.—Three energy regimes with qualitatively different spectral statistics can be identified:

$D1$: $0\le\omega\ll E_c$. This regime corresponds to times larger than the diffusion time across the system E_c^{-1}. The $q=0$ mode [cf. Eq. (7)] dominates, yielding

$$Y_2(\omega)=\frac{\Delta^2}{\pi^2}\frac{\gamma^2-\omega^2}{(\gamma^2+\omega^2)^2},$$
$$\Sigma_2(\Delta E)=\frac{1}{\pi^2}\ln\left(\frac{\Delta E^2+\gamma^2}{\gamma^2}\right), \tag{8}$$

in agreement with results for the Gaussian orthogonal ensemble of random matrix theory [3].

$D2$: $E_c\ll\omega\ll 1/\tau$. Order of $(L/l)^2$ modes contribute. Approximating the sum over $\mathbf{q}$ by an integral, one obtains [3]

$$Y_2(\omega)\simeq\frac{2(\Delta\tau)^2}{\pi}\left(\frac{L}{2\pi l}\right)^2\ln(\omega\tau),$$
$$\Sigma_2(\Delta E)\simeq\left(\frac{L}{2\pi l}\right)^2\Delta E\tau. \tag{9}$$

$D3$: $1/\tau\ll\omega$. Corresponds to classical trajectories shorter than the typical mean free path. The diagram S_2 involving only twofold impurity scattering plays the dominant role [16]. We obtain [17]

$$Y_2(\omega)=-\frac{1}{2\pi}\left(\frac{L}{2\pi l}\right)^2\frac{\Delta^2}{\omega^2},$$
$$\Sigma_2(\Delta E)=\frac{1}{2\pi}\left(\frac{L}{2\pi l}\right)^2\left[\ln(\Delta E\tau)+\frac{3}{2}\right]. \tag{10}$$

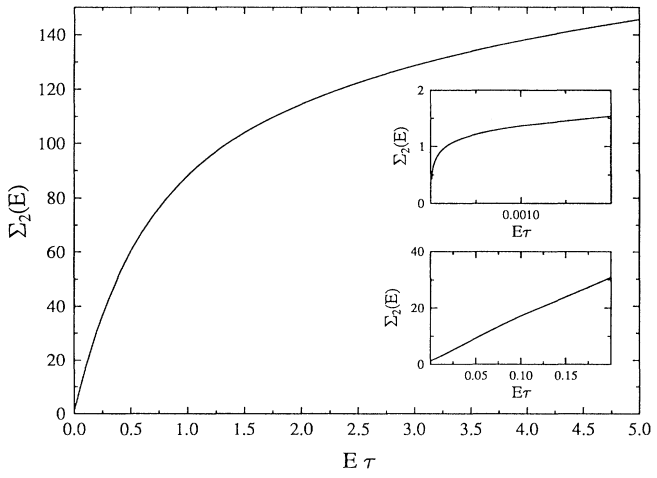

FIG. 2. $\Sigma_2(\Delta E)$ in the diffusive case ($p_f l=100$, $L/l=10$).

This logarithmic increase differs from the saturation in $\Sigma_2(\Delta E)$ which was hypothesized in Ref. [3] (cf., however, the discussion below). The above results are summarized in Fig. 2.

Ballistic systems.—There is no diffusive motion on length scales smaller than the system size, hence E_c is obsolete. Instead, the inverse time of flight $E_{tf}=v_f/L$ becomes important. Before we proceed with calculating Σ_2, we note that the disorder-averaged density of states $\langle\nu(E)\rangle$ of a ballistic sample is not constant (like in the diffusive case) but exhibits nonuniversal modulations as a function of E. The interplay between these fluctuations and the disorder induced sample to sample fluctuations discussed below will be analyzed elsewhere.

$B1$: $0\le\omega\ll 1/\tau$. On the corresponding time scales the electrons are multiply scattered, i.e., the motion is diffusive. Since $\omega\ll E_{tf}$, only the 0 mode in Eq. (7) contributes and we obtain to leading order in $\omega\tau$

$$Y_2(\omega)=\frac{\Delta^2}{\pi^2}\frac{\gamma^2-\omega^2}{(\gamma^2+\omega^2)^2},$$
$$\Sigma_2(\Delta E)=\frac{1}{\pi^2}\ln\left(\frac{\Delta E^2+\gamma^2}{\gamma^2}\right), \tag{11}$$

like in the ergodic regime for diffusive samples.

$B2$: $1/\tau\ll\omega\ll E_{tf}$. This regime corresponds to trajectories shorter than the scattering length but longer than the system size. In this regime the motion is indeed ballistic. Still the $q=0$ mode dominates and we obtain

$$Y_2(\omega)=\frac{3\Delta^2}{2\pi^2\tau^2}\frac{1}{\omega^4},$$
$$\Sigma_2(\Delta E)=-\frac{2}{\pi^2}\ln(\gamma\tau)-\frac{1}{2\pi^2}\frac{(\Delta E\tau)^2}{1+(\Delta E\tau)^2}. \tag{12}$$

The most remarkable feature of this result is the nonmonotonicity of Σ_2 as a function of energy, which implies that the fluctuations in the number of levels contained within an energy interval of width $1/\tau<\Delta E<E_{tf}$ *decrease* upon increasing the width of the energy interval.

$B3$: $\omega>E_{tf}$. As soon as ω becomes comparable with E_{tf}, $q\neq 0$ modes contribute to the sum in Eq. (7). Replacing the summation by an integral we obtain an asymptotic behavior

$$Y_2(\omega)\overset{\omega\gg E_{tf}}{\longrightarrow}-\frac{1}{2\pi}\left(\frac{L}{2\pi l}\right)^2\left(\frac{\Delta}{\omega}\right)^2,$$
$$\Sigma_2(\Delta E)\overset{\Delta E\gg E_{tf}}{\longrightarrow}\frac{1}{\pi}\left(\frac{L}{2\pi l}\right)^2\ln(\Delta E\tau). \tag{13}$$

In Fig. 3 we have plotted the analytically calculated Eq. (7) (approximating the summation by an integration) and the numerically evaluated sum. The analytical approximation turns out to blur fluctuations on a scale $\sim E_{tf}$. Qualitatively, the onset of fluctuations can be un-

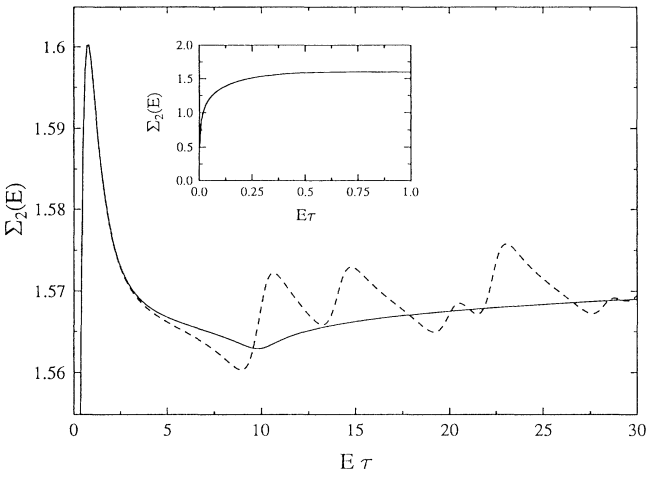

FIG. 3. $\Sigma_2(\Delta E)$ in the ballistic case ($p_f l = 10^4$, $L/l = 0.1$).

derstood within the semiclassical picture: In the regime of energies larger than the inverse scattering time only the diagram S_2 involving twofold impurity scattering contributes significantly to the correlation functions, i.e., $Y_2(t < \tau)$ is given by the sum of all periodic orbits connecting two given impurities. If $t < t_f$ there is only one periodic orbit, namely, the one connecting the impurities directly. Whenever t increases by an amount $\sim t_f$, additional orbits involving boundary scatterings become accessible and $Y_2(t)$ should increase rapidly. This time dependent structure survives Fourier transforming and leads to a fluctuating pattern in $Y_2(\omega)$. Similar phenomena can be observed in the large energy behavior of correlation functions describing clean chaotic systems as well [18].

Nonuniversality.— We have shown here that the spectral correlation function exhibits nonuniversal behavior over large energy scales. How does this behavior depend on the microscopic features of the scattering potential? To address this question, we have generalized the white noise disorder potential to a random potential with finite correlation length a. For small energies corresponding to diffusive time scales, the modification of the potential does not affect the spectral statistics qualitatively. The situation is different in the nonuniversal large energy regimes. In the case of a soft scattering potential ($p_f a \gg 1$) we find that $\Sigma_2(E)$ approaches a constant value instead of increasing logarithmically as it does in the white noise scenario. Details of our analysis will be published elsewhere.

We have enjoyed useful discussions with N. Argaman, E. Doron, Y. Imry, A. Kamenev, M. Sieber, and U. Smilansky. A.A. is a recipient of a Minerva fellowship. This work has been supported by the German-Israel Foundation for Scientific Research (GIF) and by the U.S.-Israel Binational Science Foundation (BSF).

*Present address: Institut für Theoretische Physik, Zülpicherstrasse 77, D-50937 Köln, Germany.

[1] L. P. Gor'kov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. **48**, 1407 (1965) [Sov. Phys. JETP **21**, 940 (1965)].
[2] Y. Imry, Europhys. Lett. **1**, 249 (1986).
[3] B. L. Al'tshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[4] P. Carr, J. T. Chalker, and K. A. Benedict, Ann. Phys. (N.Y.) **194**, 1 (1989).
[5] A. Schmid, Phys. Rev. Lett. **66**, 80 (1991); B. L. Altshuler, Y. Gefen, and Y. Imry, Phys. Rev. Lett. **66**, 88 (1991); F. von Oppen and E. K. Riedel, Phys. Rev. Lett. **66**, 84 (1991).
[6] S. Oh, A. Yu. Zyuzin, and R. A. Serota, Phys. Rev. B **44**, 8858 (1991); R. A. Serota and A. Y. Zyusin (to be published).
[7] B. L. Altshuler, Y. Gefen, Y. Imry, and G. Montambaux (to be published); Y. Gefen, B. Reulet, and H. Bouchiat, Phys. Rev. B **46**, 15922 (1992).
[8] A. Altland, S. Iida, A. Müller-Groeling, and H. A. Weidenmüller, Europhys. Lett. **20**, 155 (1992); Ann. Phys. (N.Y.) **219**, 148 (1992).
[9] N. Dupuis and G. Montambaux, Phys. Rev. B **43**, 14390 (1991).
[10] N. Argaman, Y. Imry, and U. Smilansky, Phys. Rev. B **47**, 4440 (1993).
[11] A. Kamenev and Y. Gefen (unpublished).
[12] H. F. Cheung, E. K. Riedel, and Y. Gefen, Phys. Rev. Lett. **62**, 587 (1989).
[13] U. Sivan and Y. Imry, Phys. Rev. B **35**, 6074 (1987).
[14] A more detailed analysis, including the study of finite magnetic fields, will be published elsewhere.
[15] B. L. Altshuler, A. G. Aronov, A. I. Larkin, and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **81**, 768 (1981) [Sov. Phys. JETP **54**, 411 (1983)].
[16] For reasons to be discussed elsewhere [14], the diagram S_1 is inessential.
[17] This result has independently been derived in a purely semiclassical formulation by N. Argaman (unpublished).
[18] M. V. Berry, Proc. R. Soc. London A **400**, 229 (1985).