String Theory Formulation of the Three-Dimensional Black Hole

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A black hole solution to three-dimensional general relativity with a negative cosmological constant has recently been found. We show that a slight modification of this solution yields an exact solution to string theory. This black hole is equivalent (under duality) to the previously discussed three-dimensional black string solution. Since the black string is asymptotically flat and the black hole is asymptotically anti–de Sitter, this suggests that strings are not affected by a negative cosmological constant in three dimensions.

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In a recent paper [1], Banados *et al.* showed that there is a black hole solution to three-dimensional general relativity with a negative cosmological constant. At first sight this is surprising, since the field equation for this theory requires that, locally, the curvature is constant. However, they showed that by identifying certain points of three-dimensional anti–de Sitter space, one obtains a solution with almost all of the usual features of a black hole. In fact, there is a two-parameter family of inequivalent identifications leading to black holes with mass M and angular momentum J. Even though the curvature is constant, the solutions have trapped surfaces, an event horizon, and nonzero Hawking temperature. When $J\neq 0$, they also have an ergosphere, and inner horizon. The solutions all approach anti–de Sitter space (without identifications) asymptotically.

This solution is easily modified to obtain an exact solution to string theory. One simply adds an antisymmetric tensor field $H_{\mu\nu\rho}$ proportional to the volume form $\epsilon_{\mu\nu\rho}$. The reason is the following. There is a well known construction (the Wess-Zumino-Witten model) for obtaining a conformal field theory describing string propagation on a Lie group. The natural metric on the group SL(2,R) is precisely the three-dimensional anti–de Sitter metric. So the Wess-Zumino-Witten (WZW) model based on SL(2,R) is an exact conformal field theory describing string propagation on anti–de Sitter space [2]. The $H_{\mu\nu\rho}$ field is required by the Wess-Zumino term and must be chosen so that the connection with torsion $H_{\mu\nu\rho}$ is flat. To obtain the black hole, one applies the orbifold procedure [3] to obtain a conformal field theory describing string propagation on the quotient space.

This solution is of interest for several reasons. An exact four-dimensional black hole in string theory has not yet been found. A few years ago, Witten showed [4] that an exact two-dimensional black hole could be obtained by gauging a one-dimensional subgroup of SL(2,R). The three-dimensional black hole has a number of advantages over the two-dimensional one. First, strings in three dimensions resemble higher-dimensional solutions in that there are an infinite number of propagating modes. One can thus examine their effect on Hawking evaporation. Second, the construction is even simpler than the two-dimensional black hole. One merely quotients by a discrete subgroup rather than gauging a continuous one. In three dimensions, there will presumably be a tachyon, which can be removed by considering the supersymmetric WZW model.

One of the most interesting properties of string theory is that different spacetime geometries can correspond to equivalent classical solutions. We will show that the three-dimensional black hole is equivalent to the charged black string solution discussed earlier [5]. Some implications of this equivalence for spacetime singularities and the cosmological constant problem will be considered.

We begin by reviewing the black hole solution discovered by Banados *et al.* [1]. (See also [6].) Anti–de Sitter space can be represented

$$ds^2=\left(1-\frac{\hat r^2}{l^2}\right)d\hat t^2+\left(\frac{\hat r^2}{l^2}-1\right)^{-1}d\hat r^2+\hat r^2 d\hat\varphi^2\,, \qquad (1)$$

where $\hat t$ and $\hat\varphi$ take any real value. If we identify $\hat\varphi=\hat\varphi+2\pi$, (1) describes a black hole. The surfaces of constant $\hat t$ and $\hat r<l$ are now compact trapped surfaces. One could also identify $\hat\varphi$ with a period other than 2π. It turns out that this corresponds to changing the mass of the black hole. This is analogous to the fact that in the absence of a cosmological constant, the mass (in three dimensions) is related to the deficit angle at infinity [7]. To add angular momentum, one periodically identifies a linear combination of $\hat\varphi$ and $\hat t$, rather than $\hat\varphi$ itself.

To be more explicit, choose two constants r_+,r_- and introduce new coordinates

$$\begin{aligned}\hat t&=(r_+t/l)-r_-\varphi\,,\\ \hat\varphi&=(r_+\varphi/l)-(r_-t/l^2)\,,\\ \hat r^2&=l^2(r^2-r_-^2)/(r_+^2-r_-^2)\,.\end{aligned} \qquad (2)$$

Then the metric (1) becomes [1]

$$ds^2=\left(M-\frac{r^2}{l^2}\right)dt^2-J dt\,d\varphi+r^2d\varphi^2+\left(\frac{r^2}{l^2}-M+\frac{J^2}{4r^2}\right)^{-1}dr^2\,, \qquad (3)$$

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where the constants M and J are related to $r_\pm$ by

$$M=\frac{r_+^2+r_-^2}{l^2},\quad J=\frac{2r_+r_-}{l}. \tag{4}$$

Identifying φ with $\varphi+2\pi$ yields a two-parameter family of black holes. By paying careful attention to the surface terms in a Hamiltonian analysis [1], one finds that M is the mass and J is the angular momentum of the solution. In general, there are two horizons where $\nabla_\mu r$ becomes null, which are located at $r=r_\pm$. These two horizons coincide when $|J|=Ml$, which is the extremal limit. Anti-de Sitter space in its usual form is recovered when $M=-1$ and $J=0$. The Killing vector $\partial/\partial t$ becomes null at $r^2=Ml^2$, which lies outside the event horizon $r=r_+$ when $J\neq 0$. This is similar to the ergosphere in the Kerr solution. Physically, it means that an observer cannot remain at rest with respect to infinity when she is close to the horizon.

What is the spacetime like near $r=0$? Since the curvature is constant, $R_{\mu\nu}=-(2/l^2)g_{\mu\nu}$, there cannot be a curvature singularity. When $J=0$ and $M>0$, the φ translation symmetry has a fixed point. This causes the solution, near $r=0$, to resemble the Taub-NUT solution and have incomplete null geodesics. However, when $J\neq 0$, the symmetry has no fixed points and the spacetime is completely nonsingular. This is consistent with the singularity theorems [8] even though the spacetime has trapped surfaces and satisfies the strong energy condition, because there are closed timelike curves. (The continuation past $r=0$ consists of r^2 becoming negative, so φ becomes timelike.) Banados *et al.* argue that one should end the spacetime at $r=0$, which avoids the causality problem but creates incomplete geodesics. However, this appears very unnatural. The four-dimensional Kerr solution also has closed timelike curves inside the inner horizon. (Although the vector $\partial/\partial\varphi$ is timelike only for $r<0$ in Kerr, there are closed timelike curves through every point with $r<r_-$. It is likely that a similar result holds here.) The closed timelike curves are not expected to produce a physical violation of causality because of the instability of the inner horizon. String theory provides another reason for not ending the spacetime at $r=0$. The WZW construction clearly includes all regions of the spacetime, including $r^2<0$.

The Hawking temperature for this black hole is

$$T=(r_+^2-r_-^2)/2\pi r_+l^2. \tag{5}$$

(The factor of l^2 was omitted in [1].) One can show that this implies that the black hole does not evaporate completely in finite time. This will be discussed elsewhere.

We now turn to string theory. We first consider these black holes in the context of the low energy approximation, and then consider the exact conformal field theory. In three dimensions, the low energy string action is

$$S=\int d^3x\sqrt{-g}\,e^{-2\phi}\left[\frac{4}{k}+R+4(\nabla\phi)^2-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]. \tag{6}$$

The equations of motion which follow from this action are

$$R_{\mu\nu}+2\nabla_\mu\nabla_\nu\phi-\tfrac{1}{4}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma}=0\,, \tag{7a}$$

$$\nabla^\mu(e^{-2\phi}H_{\mu\nu\rho})=0\,, \tag{7b}$$

$$4\nabla^2\phi-4(\nabla\phi)^2+\frac{4}{k}+R-\frac{1}{12}H^2=0\,. \tag{7c}$$

A special property of three dimensions is that $H_{\mu\nu\rho}$ must be proportional to the volume form $\epsilon_{\mu\nu\rho}$. If we assume $\phi=0$, then (7b) yields $H_{\mu\nu\rho}=(2/l)\epsilon_{\mu\nu\rho}$, where l is a constant with dimensions of length. Substituting this form of H into (7a) yields $R_{\mu\nu}=-(2/l^2)g_{\mu\nu}$, which is exactly Einstein's equation with a negative cosmological constant. The dilaton equation (7c) will also be satisfied provided $k=l^2$. Thus every solution to three-dimensional general relativity with negative cosmological constant is a solution to low energy string theory with $\phi=0$, $H_{\mu\nu\rho}=(2/l)\epsilon_{\mu\nu\rho}$, and $k=l^2$. In particular, the two-parameter family of black holes (3) is a solution with

$$B_{\varphi t}=r^2/l,\quad \phi=0\,, \tag{8}$$

where $H=dB$. An earlier argument [9] claiming that three-dimensional black hole solutions to (7) do not exist assumed that $H_{\mu\nu\rho}=0$ [10].

We now consider the dual of this solution. Duality is a well known symmetry of string theory that maps any solution of the low energy string equations (7) with a translational symmetry to another solution. (Under certain conditions, the two solutions correspond to equivalent conformal field theories [11,12].) Given a solution $(g_{\mu\nu}, B_{\mu\nu},\phi)$ that is independent of one coordinate, say x, then $(\tilde g_{\mu\nu},\tilde B_{\mu\nu},\tilde\phi)$ is also a solution where [13]

$$\begin{aligned}&\tilde g_{xx}=1/g_{xx},\quad \tilde g_{xa}=B_{xa}/g_{xx}\,,\\ &\tilde g_{\alpha\beta}=g_{\alpha\beta}-(g_{x\alpha}g_{x\beta}-B_{x\alpha}B_{x\beta})/g_{xx}\,,\\ &\tilde B_{xa}=g_{xa}/g_{xx},\quad \tilde B_{\alpha\beta}=B_{\alpha\beta}-2g_{x[\alpha}B_{\beta]x}/g_{xx}\,,\\ &\tilde\phi=\phi-\tfrac{1}{2}\ln g_{xx}\,,\end{aligned} \tag{9}$$

and α,β run over all directions except x.

Applying this transformation to the φ translational symmetry of the black hole solution (3) and (8) yields

$$\begin{aligned}\widetilde{ds}^2=&\left(M-\frac{J^2}{4r^2}\right)dt^2+\frac{2}{l}dt\,d\varphi+\frac{1}{r^2}d\varphi^2\\ &+\left(\frac{r^2}{l^2}-M+\frac{J^2}{4r^2}\right)^{-1}dr^2\,,\\ \tilde B_{\varphi t}=&-J/2r^2,\quad \tilde\phi=-\ln r\,.\end{aligned} \tag{10}$$

To better understand this solution, we diagonalize the metric. Let

$$t=\frac{l(\hat x-\hat t)}{(r_+^2-r_-^2)^{1/2}},\quad \varphi=\frac{r_+^2\hat t-r_-^2\hat x}{(r_+^2-r_-^2)^{1/2}},\quad r^2=l\hat r\,. \tag{11}$$

Then the solution becomes

$$\tilde{ds}^2=-\left(1-\frac{\mathcal{M}}{\hat{r}}\right)d\hat{t}^2+\left(1-\frac{Q^2}{\mathcal{M}\hat{r}}\right)d\hat{x}^2+\left(1-\frac{\mathcal{M}}{\hat{r}}\right)^{-1}\left(1-\frac{Q^2}{\mathcal{M}\hat{r}}\right)^{-1}\frac{l^2d\hat{r}^2}{4\hat{r}^2}, \quad (12)$$
$$\tilde{\phi}=-\tfrac{1}{2}\ln\hat{r}l,\quad \tilde{B}_{\hat{x}\hat{t}}=Q/\hat{r},$$

where $\mathcal{M}=r_+^2/l$ and $Q=J/2$. This is precisely the previously studied three-dimensional charged black string solution [5]. Notice that the charge of the black string is proportional to the angular momentum of the black hole. The horizons of the black string are at the same location as the black hole $r^2=r_\pm^2$. Since φ is periodic, both $\hat{t}$ and $\hat{x}$ will in general be periodic. To avoid closed timelike curves, one must go to the covering space.

Since the dual of the black hole is the black string, it must be possible to dualize the black string and recover the black hole. This is a little puzzling since it has been shown [14] that if you dualize (12) on $\hat{x}$, you obtain a boosted uncharged black string. The charge Q is dual to the momentum in the symmetry direction $P_{\hat{x}}$. However, one can apply duality to any translational symmetry $\partial/\partial\hat{x}+\alpha\,\partial/\partial\hat{t}$. If $\alpha<1$, then the dual is again a charged black string. If $\alpha=1$ the result is different. The Killing vector $\partial/\partial\hat{x}+\partial/\partial\hat{t}$ has norm $(\mathcal{M}^2-Q^2)/\mathcal{M}\hat{r}$, so it is spacelike everywhere but asymptotically null. One can easily verify that the dual of the black string (12) with respect to this symmetry is precisely the three-dimensional black hole.

We now consider a few special cases. The dual of the nonrotating black hole, $J=0$ and $M>0$, is the uncharged black string. This is simply the two-dimensional black hole cross S^1. For the zero mass solution $(M=J=0)$ and the extremal limit $(|J|=Ml)$ the dual is still given by (10), but the transformation to the black string breaks down. The duals are not the zero mass and extremal black string, but rather these solutions superposed with a plane fronted wave. Setting $M=J=0$ in (10), and introducing new coordinates $t=-v$, $\varphi=ul/2$, $r=e^{\hat{r}/l}$, yields

$$\tilde{ds}^2=-du\,dv+d\hat{r}^2+\tfrac{1}{4}l^2e^{-2\hat{r}/l}du^2. \quad (13)$$

This is a plane fronted wave in the presence of a dilaton [14]. Setting $J^2=M^2l^2$ in (10), and introducing new coordinates $t=-v/M$, $\varphi=l(v+Mu)/2$, $r^2=l\hat{r}$, the metric becomes

$$\tilde{ds}^2=-\left(1-\frac{Ml}{2\hat{r}}\right)du\,dv+\frac{l^2d\hat{r}^2}{4(\hat{r}-Ml/2)^2}+\frac{M^2l}{4\hat{r}}du^2. \quad (14)$$

This describes a wave of constant amplitude traveling along an extremal black string [15]. Finally, recall that the full anti–de Sitter space corresponds to $J=0$ and $M=-1$. Inserting these values into (10) and setting $t=\hat{t}+\varphi/l$ yields

$$\tilde{ds}^2=-d\hat{t}^2+\left(1+\frac{r^2}{l^2}\right)^{-1}dr^2+\left(\frac{r^2+l^2}{r^2l^2}\right)d\varphi^2, \quad (15)$$

which is the product of time and the dual of the two-dimensional Euclidean black hole.

We now turn to the exact conformal field theory description of the black hole. As mentioned earlier, this is in terms of the SL(2,R) WZW model at level k. For the noncompact group SL(2,R), k is not required to be an integer. The central charge is $c=3k/(k-2)$, so $c=26$ when $k=52/23$. One can also consider larger values of k and take the product of this black hole with an internal conformal field theory. In terms of the group, translations of $\hat{\varphi}$ in (1) correspond to the axial symmetry

$$\delta g=\epsilon\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}g+g\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right], \quad (16)$$

while translations of $\hat{t}$ correspond to the vector symmetry

$$\delta g=\epsilon\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}g-g\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]. \quad (17)$$

The general black hole is obtained by quotienting under a discrete subgroup of a linear combination of these symmetries. This can be carried out by the standard orbifold construction [3].

There is a close connection between the two- and three-dimensional black holes in string theory. Witten has shown [4] that the two-dimensional black hole can be obtained by starting with the SL(2,R) WZW model and gauging the axial symmetry (16). If one gauges the vector symmetry, one obtains the dual of the black hole, which turns out to have the same geometry. One cannot gauge a general linear combination of the symmetries because of an anomaly.

Roček and Verlinde have shown [11] that for a positive definite target space having a spacelike symmetry with compact orbits, the low energy duality (9) corresponds to an equivalence between exact conformal field theories. For Lorentzian target spaces, equivalence has not yet been rigorously established. In addition to the obvious difficulty of convergence of the functional integral, there are other issues involving potential closed timelike curves. Nevertheless, since the equivalence does not explicitly depend on the signature of the target space, one expects it to hold in this case also. The effect of duality on a WZW model has been investigated [11,16]. If one dualizes with respect to an axial or vector symmetry, the result (after a simple shift of coordinate) is just the product of U(1) and the WZW model with this symmetry gauged [11]. This explains why the dual of the nonrotating black hole is just the product of the two-dimensional black hole and U(1). It also explains why the dual of anti–de Sitter is the product of time and the dual of the Euclidean black hole. However, we have seen that the dual of the general rotating black hole is the charged black string which is not a simple product. The exact conformal field theory associated with this solution is also known [5]. One starts with the group SL(2,R)×U(1) and gauges the axial symmetry (16) of SL(2,R) together with rotations of U(1).

Solutions to the low energy field equations cannot be trusted in regions of large curvature. But since the three-dimensional black hole has constant curvature (which is small for large k), it should be a good approximation everywhere. In fact, for a WZW model, the exact metric differs from the low energy approximation only by an overall rescaling [17,18]. So the exact three-dimensional black hole metric is simply proportional to (3), and (for $J\neq 0$) is nonsingular (although the stability of the inner horizon and the effect of the closed timelike curves remain to be investigated). A candidate for the exact two-dimensional black hole metric has been found [17, 19,20]. It has recently been shown [21] that this metric is also free of curvature singularities (although the dilaton diverges inside the event horizon). This can be viewed as increasing evidence that exact black holes in string theory do not have curvature singularities. But the evidence is far from conclusive.

A candidate for the exact three-dimensional black string metric has also been found [18,22], which does have a curvature singularity. However, the fact that the solution is equivalent to one without a curvature singularity suggests that it may also correspond to a nonsingular conformal field theory.

Perhaps the most remarkable consequence of the equivalence between the black hole and black string comes from the fact that the black hole is asymptotically anti–de Sitter while the black string is asymptotically flat (although the dilaton grows linearly at infinity). Since they are equivalent, it suggests that a negative cosmological constant has no effect on strings in three dimensions. The reason that the asymptotic structure of the spacetime changes under duality is that the length of the circles parametrized by φ does not approach a constant at infinity. This phenomenon has been noticed before, but in previous examples the asymptotic behavior of the dual space did not have a simple physical interpretation. For example, consider four-dimensional Minkowski spacetime $ds^2=-dt^2+dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$. If we dualize on φ the metric is identical except that $g_{\varphi\varphi}$ is changed to $(r^2\sin^2\theta)^{-1}$, which is singular along the axis $\theta=0,\pi$. The three-dimensional black hole seems to be the first example in which two different asymptotic behaviors each have a simple physical interpretation.

This suggests a novel resolution of the cosmological constant problem. Perhaps a solution with a cosmological constant in string theory is equivalent to one without. Strings may not be affected by a cosmological constant. Unfortunately, a straightforward generalization of our results to higher dimensions does not relate a solution with a cosmological constant to one without. One can start with the charged black string in D dimensions [23] which is asymptotically flat, and dualize with respect to a symmetry that is spacelike but asymptotically null. The result is a metric which is neither asymptotically flat nor asymptotically anti–de Sitter. However, given our elementary understanding of duality in string theory, one cannot rule out the possibility that this effect will play a role in resolving the cosmological constant problem.

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