Multidimensional Solitons in Quadratic Nonlinear Media

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(Received 21 June 1993)

A new type of optical soliton is presented for two-wave interactions in quadratic nonlinear media. Unlike the conventional soliton in cubic nonlinear media, the most unique feature of the present solitons is found in the feasibility of multidimensional confinement. A compact analytical expression for the new solitons with arbitrary transverse dimension is derived through the self-consistent-field approximation. With available nonlinear materials in mind, a specific method for observing the multidimensional solitons is investigated.

PACS numbers: 42.50.Rh, 02.60.Lj, 42.60.Jf, 42.65.Jx

The concept of solitons has now become ubiquitous in modern sciences, and indeed can be found in various branches of physics [1]. In optics, the formation, propagation, and detection of solitons are of topical interest, not only in their curious phenomenology but in diverse potential applications to long-haul communication and ultrafast signal-routing systems [2]. Algebraically, these optical solitons are a particular solution of the $(1+1)$-dimensional nonlinear Schrödinger equation (NLSE) with cubic (third-order: $\chi^{(3)}$) nonlinearity. Here the symbol $(D+1)$ indicates the D-dimensional confinement $(D=1,2,3)$ for the transverse axes with a single longitudinal (propagation) axis. As is well known, there exist two kinds of solitons in the $(1+1)$-dimensional NLSE: bright and dark solitons [2]. In addition to the $(1+1)$-dimensional NLSE, analytical and numerical studies of bright field solutions of the higher-dimensional $(D\geq 2)$ NLSE's have been made [3-5]. However, in contrast to the canonical $(1+1)$-dimensional NLSE, no solitonic solution has been found but only stationary solitary-wave solutions, which have been found to be extremely unstable [5], have been obtained. This instability against fluctuations leads eventually to the blowup (the collapse) at least in the framework of scalar wave theory. One may notice that it is this unstable feature that prevents one from taking advantage of the higher-dimensional light beams as a means of signal transmissions. On the other hand, in more recent years the multidimensional dark beams in Kerr nonlinear media, such as the fundamental-dark-soliton crosses [6] and the optical-vortex solitons [7], have been studied by Swartzlander and coworkers. It should be noted here that, as suggested concerning the macroscopic cascading of $\chi^{(2)}$ nonlinearities to obtain nonlinear phase shifts [8], the quadratic nonlinearity may be useful for forming solitons. In this Letter we present a novel type of optical (envelope) soliton, taking advantage of mutual guiding assistance resulting from two-wave parametric interactions in quadratic (second-order: $\chi^{(2)}$) nonlinear media [9]. Unlike the conventional soliton that is based upon the cubic $(\chi^{(3)})$ nonlinearity, the most unique and fascinating feature of the present $\chi^{(2)}$ solitons lies in the feasibility of the multidimensional confinement for all transverse axes. This could result also in a stable light-bullet formation, which has been impossible for pulsed beams propagating in a cubic nonlinear medium because of its supercritical instability [5,10]. A compact analytical expression for the new solitons with arbitrary transverse dimension is derived with the aid of self-consistent-field (Hartree-like) approximation. With available nonlinear optical materials in mind, a specific method for observing the multidimensional solitons is presented. It is found that the intensity threshold can be lowered optionally by tuning the temperature.

We consider a phase-matched traveling-wave configuration of optical wave mixing between the fundamental (ω) and the second-harmonic (SH: 2ω) frequency components through the quadratic nonlinearity of a dielectric medium. From Maxwell's and material equations with a propagation factor, $\exp[in(\beta z-\omega t)]$ $(n=1,2)$, being implied, the coupled wave equations of the slowly varying electric-field amplitudes, $A(\omega)$ and $A'(2\omega)$, along the propagation axis z are derivable [4,11]:

$$-i2B\partial_\zeta A=\rho^{1-D}\partial_\rho(\rho^{D-1}\partial_\rho A)+(\varepsilon-B^2)A+\kappa A^*A', \tag{1a}$$

$$-i4B\partial_\zeta A'=\rho^{1-D}\partial_\rho(\rho^{D-1}\partial_\rho A')+4(\varepsilon'-B^2)A'+\kappa' A^2, \tag{1b}$$

where $\zeta=k_0z$ (k_0 being the wave number of the fundamental wave in a vacuum); $\rho=k_0x$ for $D=1$ $(-\infty<x<\infty$, with x being a transverse axis); $\rho=k_0r$ for $D=2,3$ $(0\leq r<\infty$, with r being the radial axis, which includes both space and time for $D=3)$; $B=\beta/k_0$ (β being a phase constant along the z axis); ε is the relative permittivity; κ represents a relevant component d_{ij} $(i=1,2,3;\ j=1,2,\ldots,6)$ that is involved in the quadratic nonlinear tensor $[d]$; and the asterisk denotes complex conjugate. In this Letter the prime (no prime) indicates the quantity relevant to the SH (the fundamental) frequency component [e.g., $\varepsilon=\varepsilon(\omega)$, $\varepsilon'=\varepsilon(2\omega)$]. In the derivation of Eq. (1) we have confined ourselves to the nodeless, symmetric beam profile in a transparent medium; effects due to absorptions will be mentioned later.

0031-9007/93/71(20)/3275(4)$06.00

For dissipation-free systems the permutation symmetry [4] requires that $\kappa = \kappa'$.

First we concentrate our attention on the one-dimensional confinement [$D=1$ in Eq. (1)]. Imposing the stationarity, $\partial_\zeta \equiv 0$, $\partial_\zeta \equiv 0$, reduces Eq. (1) to

$$d_\rho^2 A + (\varepsilon - B^2) A + \kappa A^* A' = 0\,, \tag{2a}$$

$$d_\rho^2 A' + 4(\varepsilon' - B^2) A' + \kappa' A^2 = 0\,. \tag{2b}$$

In an effort to obtain an exact stationary solution, as an ansatz we set

$$A(\rho) = A_0 \operatorname{sech}^2(\alpha\rho)\,, \tag{3a}$$

$$A'(\rho) = A_0' \operatorname{sech}^2(\alpha\rho)\,, \tag{3b}$$

where A_0, A_0', and α are unknown real parameters to be determined below. On substitution of Eq. (3) into Eq. (2), one obtains

$$\varepsilon - B^2 + 4\alpha^2 = 0\,, \tag{4a}$$

$$4(\varepsilon' - B^2) + 4\alpha^2 = 0\,, \tag{4b}$$

$$\kappa A_0 A_0' - 6\alpha^2 A_0 = 0\,, \tag{4c}$$

$$2\kappa' A_0^2 - 6\alpha^2 A_0' = 0\,, \tag{4d}$$

which can be solved for B, α, A_0, and A_0',

$$B = (\varepsilon' + \tfrac{1}{3}\Delta\varepsilon)^{1/2}\,, \tag{5a}$$

$$\alpha = (\tfrac{1}{3}\Delta\varepsilon)^{1/2}\,, \tag{5b}$$

$$A_0 = \pm [2/(\kappa\kappa')]^{1/2} \Delta\varepsilon\,, \tag{5c}$$

$$A_0' = (2/\kappa)\Delta\varepsilon\,, \tag{5d}$$

with $\Delta\varepsilon \equiv \varepsilon' - \varepsilon$.

Subsequently, we consider the general case with arbitrary dimension. Unlike the case of the one dimension ($D=1$), for arbitrary dimension, the exact analytical approach is unavailable. Below we shall obtain approximate stationary solution through the use of a self-consistent-field (a Hartree-like) approach [12–14]. With this method the first step is to assume an ansatz in the form of a separation of variables,

$$A(x_1, x_2, \ldots, x_D) = f(x_1) f(x_2) \cdots f(x_D)\,, \tag{6a}$$

$$A'(x_1, x_2, \ldots, x_D) = f'(x_1) f'(x_2) \cdots f'(x_D)\,, \tag{6b}$$

where $\sum_{j=1}^D x_j^2 = \rho^2$.

Substituting Eq. (6) into Eq. (1a) and multiplying both sides of the resultant equation by $\prod_{j\neq i}^D f^*(x_j)$, we obtain an integrodifferential equation,

$$d_{x_i}^2 f + (\tilde{\varepsilon} - \tilde{B}^2) f + \tilde{\kappa} f^* f' = 0\,, \tag{7}$$

with a modified relative dielectric constant and nonlinearity, respectively,

$$\tilde{\varepsilon} = \varepsilon - (D-1) \int_{-\infty}^{\infty} |d_u f|^2 du \bigg/ \int_{-\infty}^{\infty} |f|^2 du\,, \tag{8a}$$

$$\tilde{\kappa} = \left(\int_{-\infty}^{\infty} f^{*2} f' du \bigg/ \int_{-\infty}^{\infty} |f|^2 du \right)^{D-1} \kappa\,. \tag{8b}$$

Similarly, substituting Eq. (6) into Eq. (1b) and multiplying both sides by $\prod_{j\neq i}^D f'^*(x_j)$, we obtain

$$d_{x_i}^2 f' + 4(\tilde{\varepsilon}' - \tilde{B}^2) f' + 2\tilde{\kappa}' f^2 = 0\,, \tag{9}$$

with

$$\tilde{\varepsilon}' = \varepsilon' - \tfrac{1}{4}(D-1) \int_{-\infty}^{\infty} |d_u f'|^2 du \bigg/ \int_{-\infty}^{\infty} |f'|^2 du\,, \tag{10a}$$

$$\tilde{\kappa}' = \left(\int_{-\infty}^{\infty} f'^* f^2 du \bigg/ \int_{-\infty}^{\infty} |f'|^2 du \right)^{D-1} \kappa'\,. \tag{10b}$$

Here one finds Eqs. (7) and (9) formally identical, respectively, to Eqs. (2a) and (2b). Thus, as an explicit ansatz one can set the one similar to Eq. (3),

$$f(x_j) = f_0 \operatorname{sech}^2(\alpha x_j)\,, \tag{11a}$$

$$f'(x_j) = f_0' \operatorname{sech}^2(\alpha x_j)\,, \tag{11b}$$

for $j = 1, 2, \ldots, D$, where f_0, f_0', and α are unknown parameters to be determined in a self-consistent fashion. As in Eq. (5) we obtain

$$\tilde{B} = (\tilde{\varepsilon}' + \tfrac{1}{3}\Delta\tilde{\varepsilon})^{1/2}\,, \tag{12a}$$

$$\tilde{\alpha} = (\tfrac{1}{3}\Delta\tilde{\varepsilon})^{1/2}\,, \tag{12b}$$

$$A_0 = \pm [2/(\tilde{\kappa}\tilde{\kappa}')]^{1/2} \Delta\tilde{\varepsilon}\,, \tag{12c}$$

$$A_0' = (2/\tilde{\kappa})\Delta\tilde{\varepsilon}\,, \tag{12d}$$

with $\Delta\tilde{\varepsilon} \equiv \tilde{\varepsilon}' - \tilde{\varepsilon}$, $A_0 \equiv f_0^D$, and $A_0' \equiv f_0'^D$. Substitution of Eq. (11) into Eqs. (8) and (10) leads to

$$\tilde{\varepsilon} = \varepsilon - \tfrac{4}{5}(D-1)\tilde{\alpha}^2\,, \tag{13a}$$

$$\tilde{\kappa} = (4f_0'/5)^{D-1}\kappa\,, \tag{13b}$$

$$\tilde{\varepsilon}' = \varepsilon' - \tfrac{1}{5}(D-1)\tilde{\alpha}^2\,, \tag{13c}$$

$$\tilde{\kappa}' = (4f_0^2/5f_0')^{D-1}\kappa'\,. \tag{13d}$$

From Eqs. (13a) and (13c), for $D < 6$, we obtain

$$\Delta\tilde{\varepsilon} = [5/(6-D)]\Delta\varepsilon\,. \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (12), we finally arrive at the explicit analytical expression of the beam parameters with arbitrary dimension D ($D<6$):

$$\tilde{B} = B\,, \tag{15a}$$

$$\tilde{\alpha} = \{5\Delta\varepsilon/[3(6-D)]\}^{1/2}\,, \tag{15b}$$

$$A_0 = \pm [2^{(5-4D)/2} 5^D/(6-D)] \Delta\varepsilon/(\kappa\kappa')^{1/2}\,, \tag{15c}$$

$$A_0' = [2^{3-2D} 5^D/(6-D)] \Delta\varepsilon/\kappa\,. \tag{15d}$$

It is interesting to note that the phase constant B is independent of the confinement dimension. With the width parameter $\tilde{\alpha}$ being evaluated by Eq. (15b), the intensity

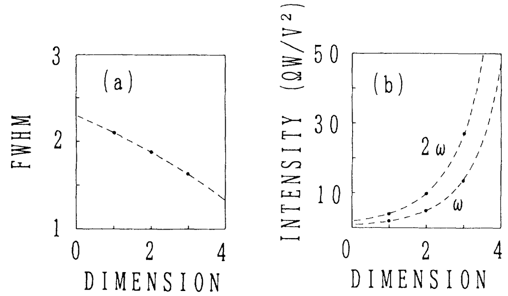

FIG. 1. Soliton shape parameters as a function of transverse dimension. (a) Normalized spot size that is defined by $(\Delta\varepsilon)^{1/2}k_0$ FWHM and (b) normalized peak intensity that is defined by $[\kappa\kappa'/(\Delta\varepsilon)^2]A_0^2$ for the fundamental component (ω), and by $(\kappa/\Delta\varepsilon)^2 A_0'^2$ for the SH component (2ω). The definition of the symbols is given in the text.

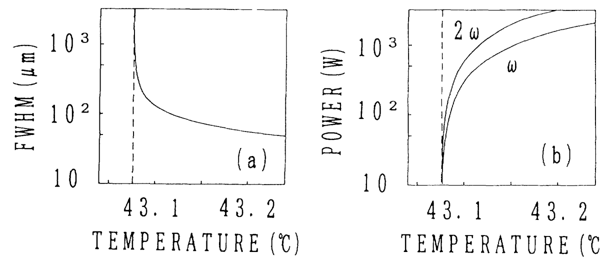

FIG. 2. Temperature dependence of soliton beam parameters ($D=2$, $\lambda=1.064$ μm). (a) Intensity FWHM and (b) optical power. The dashed line indicates the critical temperature, $t_{cr}=43.076\,°$C, at which the permittivity detuning $\Delta\varepsilon$ between the two spectra vanishes completely.

FWHM can be calculated by the relation FWHM $=1.212/k_0\tilde{\alpha}$. For $D=1$, Eqs. (15b)–(15d) are reduced to Eqs. (5b)–(5d), respectively. Typical shape parameters of the self-consistent solitary-wave solution versus the confinement dimension are plotted in Fig. 1. Although the plots are connected with the dashed line, only the three points for $D=1$, 2, and 3 are physically meaningful. As expected from Eqs. (15b)–(15d), the spot size shrinks gradually with increasing the confinement dimension, whereas the peak intensities at the beam center grow as the dimension increases.

To verify that Eqs. (1) really have a solitonic solution, and to check whether the self-consistent solution [Eq. (6) with Eqs. (11) and (15)] is a good approximate solution for the general multidimensional solitons, we have implemented numerical stability analyses of the evolution equations, Eq. (1), using an extended version of the iterative finite-element method [15]. The validity of this methodology was fully ensured through extensive applications to stability analyses of nonlinear guided waves in Kerr-like media [15]. With this method, one first inputs a stationary field into nonlinear wave equations, and traces the subsequent variation of the input field during interactions. If the input solution were stable, it would converge eventually to the exact solitonic solution. Through careful analyses we have arrived at the confirmation that at least for $D=1$, 2, and 3, the solitary-wave solution we have derived above is stable against propagation. (Similar calculations for the multidimensional cubic NLSE showed instabilities, i.e., blowup, as predicted analytically [5]). This verifies that Eqs. (1) have a solitonic solution for $D=1,2,3$ and that the self-consistent stationary solution is indeed satisfactory as an approximate analytic form of the multidimensional solitons. It should be noted here that this conclusion is consistent with that presented previously by Rasmussen and Rypdal [5], who discussed analytic properties of a single NLSE with a general power-law nonlinear term. In situations where the walk-off effect due to the group-velocity mismatch between the two components is negligible, the three-dimensional soliton ($D=3$) will permit us to observe a stable light bullet, i.e., a spatiotemporal soliton, which has been impossible for pulsed beams that propagate in a cubic nonlinear medium [5,10]. Under the same peak amplitude, comparison has been made between the eigenvalue B of Eq. (15a) with Eq. (5a) and that obtained through the numerical stability analysis with a material given below being assumed ($t=43.90\,°$C). The relative error of the approximate eigenvalue has been found to be 3.5×10^{-9} for $D=2$, and 1.3×10^{-8} for $D=3$, which indicates again that our self-consistent approach is justified.

Finally, the possibility of experimentally observing evidence for the multidimensional solitons should be mentioned. As obvious from Eqs. (15c) and (15d), to reduce the intensity threshold for forming the solitons one needs to select a material that exhibits a smaller permittivity difference between the two frequency components or larger quadratic nonlinearity. Of some candidates we have found nonlinear interaction through d_{31} in lithium niobate ($LiNbO_3$) to be the best for this purpose. In this configuration the fundamental wave is an ordinary (o) wave, whereas the SH wave is an extraordinary (e) wave. Taking advantage of the different temperature dependence of the two refractive indices (n_o,n_e) [16], one can tune the magnitude of $\Delta\varepsilon$ in an arbitrary fashion. For some typical wavelengths we have examined the variation of $\Delta\varepsilon$ $[=n_e^2(2\omega)-n_o^2(\omega)]$ on the temperature. Through the examination we have found that for $\lambda=1.064$ μm from a Nd:YAG (where YAG denotes yttrium aluminum garnet) laser, at $t_{cr}\equiv43.076\,°$C, the difference $\Delta\varepsilon$ vanishes completely. To examine the variation of soliton parameters in the vicinity of the critical temperature t_{cr} for the two-dimensional solitons ($D=2$, $\lambda=1.064$ μm), we plot in Fig. 2 the intensity FWHM and the optical powers. Here P (P') indicates the time-averaged power of the fundamental (the SH) frequency component. The vertical dashed line drawn in Fig. 2 indicates $t=t_{cr}$, and all the material data are extracted from Ref. [16]. It should be noted that only the upper-half region, $t>t_{cr}$, is allowable, and the region, $t < t_{cr}$, is forbidden because therein $\Delta\varepsilon < 0$, which results, from Eq. (15b), in the imaginary value of α. It is seen from Fig. 2(a) that the spot size of the solitons decreases with increasing temperature, but the rate gets smaller as the temperature increases. The more interesting quantity to observe in the multidimensional solitons is the optical powers that are required to sustain these during propagation. From Fig. 2(b) we find that in the close vicinity of t_{cr} the powers grow exponentially with increasing temperature. With a commercially available thermostat the temperature deviation of 0.01–0.1 °C would be achievable. For instance, at 43.09 °C, one finds from Fig. 2(b) $P = 180$ W and $P' = 360$ W, which can be realized with a currently available mode-locked technique. Surface damage would be avoidable by adopting a pulsed-mode operation [16]. Effects due to linear absorption will be negligible with propagation distance considerably shorter than the absorption length L_a. For $LiNbO_3$, from the data available [16], we estimate $L_a = 12.5$ cm for $\lambda = 1.064$ μm and $L_a' = 35.7$ cm for $\lambda' = \frac{1}{2}\lambda = 0.532$ μm. Both the two-photon absorption (TPA) of extraordinary green light (TPA coefficient being 2.9×10^{-9} cm/W [16]) and the self-focusing effect (Kerr coefficient being 2.2×10^{-21} m^2/V^2 [17]) can be ignored at least within the power scale shown in Fig. 2(b).

In conclusion, we have found a novel type of optical soliton which takes advantage of mutual guiding assistance due to two-wave mixing in quadratic nonlinear media. Unlike the conventional soliton based upon cubic nonlinearity, the most unique and fascinating feature of the new solitons has been found in the feasibility of the multidimensional confinement for all transverse axes that include both space and time.

This research was partially supported by a Scientific Research Grant-In-Aid from the Ministry of Education, Science and Culture, Japan.

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