Fundamentally Discrete Stochastic Model for Wind Ripple Dynamics

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We present a discrete, stochastic model for wind formed ripples in sand, which are observed to increase in size through mergers and seemingly approach an asymptotic spatial scale. The model is shown to predict (1) a logarithmic increase in pattern scale with time, (2) a proportionality between the microscopic discrete scale (sand grain diameter) and the macroscopic pattern scale (ripple height), and (3) a lack of scale separation. The implications are that growth and apparent stabilization of scale both can be explained by a single mechanism, and that the evolution of wind ripples and other physical systems of this type cannot be modeled by either deterministic methods or spatial continuum methods.

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Wind formed ripples in sand are among the many striking landform patterns characterized by a definite scale (Fig. 1). Sand grains on a sand bed are transported and rearranged into wind ripples primarily by high-energy impacts with the surface of grains accelerated by the wind on long, low-angle trajectories [1]. Because the impacts are roughly random and the scattering of finite size grains on the bed due to an impact has a wide distribution of outcomes [2], sand transport in ripples is inherently a discrete stochastic process. Starting from a flattened sand bed, well-formed wind ripples with a characteristic spacing have been observed to develop by coalescence and lateral organization of small sand bumps [2,3].

One key behavior of ripples promoting this organization is that smaller ripples travel faster than larger ripples. Mean ripple translation speed is inversely related to size. This is because ripple speed is the mean distance a single grain in a ripple moves per unit time, or the ratio of the sum of the distances traveled by grains in a ripple per unit time (approximately proportional to the surface area of the ripple exposed to impacts) to the number of grains in the ripple (proportional to the volume of the ripple). Differences in ripple speed can lead to mergers between ripples and a consequent progressive increase in the mean ripple size. This sequence of events has been reproduced in computer simulations based on the local physics of grain-bed impacts only, suggesting that wind ripples can form through self-organization [4–6].

Whereas previous models of wind ripple formation have ignored this initial development [1,7,8], herein we explore the hypothesis that mergers and ripple interactions play a dominant role in the development of the scale of ripples. Mergers occur when the depth of a trough separating two ripples (or "ripple height"), h (Fig. 2), vanishes. The variable h changes by a discrete amount at least as large as a sand grain diameter d as the result of a single impact on the ripple crest. If mergers are important, then the ripple height (and not the spacing of ripples, as in previous treatments) is the primary variable.

Our goal is to understand how the evolution of ripple

FIG. 1. Wind ripples, with mean spacing 10 cm, at the Algodones Dunes, California.

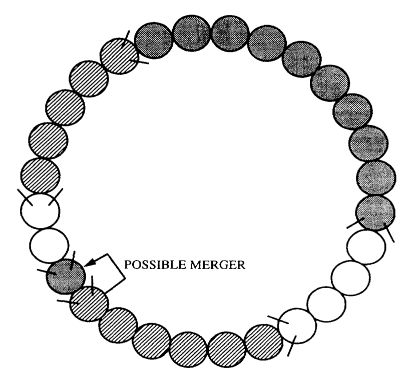

FIG. 2. Sketch of triangular ripple simulation model. Ripples translate by removing a thin slice of sand of thickness d from the upwind surface of a ripple and depositing it on the downwind side. A merger occurs when the depth of a ripple trough h vanishes.

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scale is related to the discrete stochastic interactions described above. To this end we consider a simple model of a ring composed of M circular segments, each with diameter D (see Fig. 3). By designating N of these segments as "heads," a grouping of the segments into N entities that we call "worms" is obtained. Worm i is comprised of n_i segments, and so has "size" n_i and length $x_i = Dn_i$. The worms move around the ring in real (continuous) time according to the following stochastic rule, wherein β is a positive constant: $(\beta/n_i)dt$ is the probability that the head of worm i will jump exactly one segment forward (clockwise) in the next infinitesimal time interval dt $(i=1,\ldots,N)$.

The advancement of a worm's head increases its size by 1 and simultaneously decreases the size of the worm immediately in front by 1. If the worm in front consists of only a single segment (its head), then a merger between the two worms occurs, and the number of worms on the ring decreases by 1.

The tendency in this model is toward worms with size equal to the average size $\bar{n} = M/N$, because the head of a shorter worm tends to move more frequently, which lengthens the worm, whereas the head of a longer worm tends to move less frequently, which promotes a shortening of the worm. Random fluctuations in the size of a worm about the average $\bar{n}$ are intrinsic to this model; the scale of these fluctuations will be characterized by a standard deviation σ. Extreme fluctuations result in worm mergers and a concomitant irreversible increase in $\bar{n}$. The process by which mergers occur exclusively through discrete fluctuations will be termed here stochastic merging.

Although this model does not include all of the physics of wind ripples in three dimensions, it does have four important characteristics in common with wind ripples: (1) an inverse dependence of mean translation speed on size, (2) stochastic variations in size by a discrete amount, (3) increase in mean size by mergers, and (4) a tendency toward uniform size.

We are interested in the evolution of $\bar{n}$ and σ. Mathematically, the N-worm system is an $(N-1)$-variate jump-type Markov process. An exact analytical treatment for $N>2$ would be very difficult, so we make a mean-field approximation: We focus on a single worm and assume that its trailing worm always has size $\bar{n}$. The size of the considered worm then becomes an independent univariate, "birth-death" Markov process with stepping functions [9]

$$W_+(n) = \beta/n, \quad W_-(n) = \beta/\bar{n}. \tag{1}$$

By definition, $W_\pm(n)dt$ is the probability that the size of the worm changes from its present value n to $n \pm 1$ in the next infinitesimal time interval dt.

The key to our analysis is the calculation of the average total time $t(n;\bar{n},0)$ that the worm's size has the value n during a first passage from $\bar{n}$ to 0. By using the so-called pedestrian method for calculating the mean first passage time [9,10], one can derive the recursion relations

$$t(1;\bar{n},0) = \frac{1}{W_-(1)}, \tag{2a}$$

$$t(n;\bar{n},0) = \frac{\theta(\bar{n}+1-n) + W_+(n-1)t(n-1;\bar{n},0)}{W_-(n)}, \quad n \geq 2, \tag{2b}$$

where $\theta(n)$ is the Heaviside function. In essence, Eq. (2a) follows from the fact that, in a passage from $\bar{n}$ to 0, there will be exactly one transition $1 \to 0$. Equation (2b) follows from the fact that, in a passage from $\bar{n}$ to 0, the total number of $n \to n-1$ transitions minus the total number of $n-1 \to n$ transitions will be exactly 1 if $n \leq \bar{n}$, and exactly 0 if $n > \bar{n}$. By analytically iterating these recursion relations [9,10], and then invoking Eqs. (1) and making a series of approximations for n near $\bar{n} \gg 1$ (the details of which will be described elsewhere), we obtain

$$t(n;\bar{n},0) = \frac{\bar{n}e^{\bar{n}}}{\beta} \frac{e^{-(n-\bar{n})^2/2\bar{n}}}{\sqrt{2\pi\bar{n}}}. \tag{3}$$

The average time $T(\bar{n},0)$ for the size of the worm to go from $\bar{n}$ to 0 is the sum over $t(n;\bar{n},0)$ from $n=1$ to ∞. That sum gives, to a good approximation,

$$T(\bar{n},0) = \bar{n}e^{\bar{n}}/\beta. \tag{4}$$

The ratio $t(n;\bar{n},0)/T(\bar{n},0)$ is the fraction of time that the worm has size n. This ratio is approximately normal with mean and variance $\bar{n}$; thus, the standard deviation of the worm size is $\sigma = \sqrt{\bar{n}}$. The *uniformity* of the worm pattern, as measured by the smallness of the relative standard deviation $\sigma/\bar{n} = 1/\sqrt{\bar{n}}$, is therefore high in the case $\bar{n} \gg 1$; this finding lends support to our mean-field approximation.

The single-worm first passage time from $\bar{n}$ to 0 turns out to be approximately exponentially distributed; thus, the average time before the first of the N worms disappears in a merger is approximately $T(\bar{n},0)/N$. Because a merger causes the average worm size to increase by $\bar{n}/N$ for $N \gg 1$, the average rate of increase in $\bar{n}$ is

$$\frac{d\bar{n}}{dt} = \frac{\bar{n}/N}{T(\bar{n},0)/N} = \beta e^{-\bar{n}}. \tag{5}$$

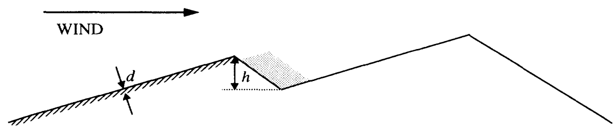

FIG. 3. Schematic for worm model. Worms are placed head to tail on a ring. In a time interval dt, the head of worm i moves forward one segment with probability $(\beta/n_i)dt$, with $\beta > 0$ and n_i the number of segments comprising the worm.

Solving for $\bar{n}(t)$, and then recalling that a mean worm size $\bar{n}$ implies a mean worm length $\bar{x}=D\bar{n}$, we conclude that the mean worm length evolves with time according to

$$\bar{x}(t)=D\ln(\beta t+e^{\bar{x}(0)/D})\,. \tag{6}$$

Exact Monte Carlo simulations of the worm model suggest that the approximations leading to Eq. (6) give reasonably accurate results for $\bar{x}\geq 5D$.

The results of this analysis have two especially interesting characteristics. First, even though $\bar{x}(t)$ in Eq. (6) increases without bound, an observer might infer that $\bar{x}(\infty)$ is finite, because the function $\ln(t)$ has the property that its graph over any finite time interval $0<t<t^*$ *appears* to be approaching a finite asymptote that lies slightly above $\ln(t^*)$. Therefore, a progressive increase in length scale and an apparent attainment of an equilibrium length scale both can arise from stochastic merging.

The second implication of our worm model analysis is that neither a deterministic approach nor a continuum approach can adequately describe the process that ultimately is responsible for the growth of the worm pattern scale, namely, mergers. A deterministic approach cannot work because mergers arise exclusively from against-the-odds fluctuations, and these are not treated in any approach that considers only average behavior. A continuum approach cannot work because the relative size of the fluctuations, being $\sigma/\bar{n}=1/\sqrt{\bar{n}}=\sqrt{D/\bar{x}}$, vanishes in the continuum limit $D/\bar{x}\rightarrow 0$. There are only two ways to achieve the continuum limit of D being vanishingly small compared to $\bar{x}(t)$ in Eq. (6): either take $\bar{x}(0)/D$ to be infinitely large, in which case the passage of time beyond $t=0$ will have no sensible effect, or else take βt to be infinitely large, in which case the dynamical phase of the problem is over. This contrasts with problems treatable by continuum mechanics, such as fluid flow, where the macroscopic behavior is insensitive to the microscopic details. This insensitivity makes possible (for example) such techniques as the lattice gas method, with "molecules" much larger than in a natural fluid [11]. By contrast, in the worm model, $\bar{x}$ is directly proportional to D, which has no macroscopic interpretation. Consequently, scale separation between $\bar{x}$ and D is effectively prevented for systems that evolve for a practical period of time. Pattern formation models based on partial differential equations [e.g., 12,13], even though some exhibit logarithmic growth, do not share this property endemic to stochastic merging.

We hypothesized earlier that the ripple height h can be identified with the length of a worm x. Therefore, the height (and the spacing, for a constant shape) of wind ripples increases logarithmically with time with a coefficient in front of the logarithm that is a small multiple of the grain diameter d. This prediction of the worm model can be tested by computer simulation and by the limited number of physical observations available.

Two-dimensional computer simulations of ripples approximated as triangular in cross section were performed in a periodic cell. The properties of this simulation algorithm were chosen to incorporate basic features of natural wind ripples, including ripple translation by stochastic sand transport and translation speed which is an inverse function of size, while keeping the model simple and its results easily interpretable. In the simulations, sand is transported and ripples are translated by removing a thin slice of sand of thickness the sand grain diameter d from the upwind surface of a ripple and depositing it on the downwind side of the ripple (Fig. 2). The probability of transport at each small time step is chosen to reproduce the proper mean translation speed. Simulated ripples merge when the trough between them disappears. The asymptotic mean ripple height $\bar{h}$ is found to be consistent with a logarithmic dependence on time, with coefficient $D\sim 1d$. In this simulation, $\bar{h}$ is approximately proportional to the mean spacing, which is plotted in Fig. 4 for comparison to observations of natural wind ripple development. For small values of $d/\bar{h}$, $\bar{h}$ is found to remain constant over a long period of time; i.e., no mergers occur. A simulation in three dimensions of ripple formation where sand is transported as individual grains (Landry and Werner, in preparation) also appears to be asymptotically consistent with the predictions of the simple worm model, with $D\sim 4d$ (Fig. 4). The values of the coefficient of the logarithm $D>d$ observed may be associated with the effect of the greater number of degrees of freedom in three dimensions.

Physical measurements of wind ripple spacing also lend support to the application of the worm model to wind ripple formation. First, data on ripple evolution in a wind tunnel [4] and in the field [2,5] are consistent (within uncertainties) with asymptotic logarithmic increase in spacing with D ranging from $1d$ to $6d$ (Fig. 4). Second, wind ripple spacing has been observed to scale roughly with grain diameter [3]. Continuum models of ripples [1,7,8] do not predict this proportionality between grain diameter and ripple crest spacing and height. Third, the crest-to-trough height of well-developed ripples typically is on the order of 10–20 grain diameters, in agreement with the prediction that scale separation is lacking.

In summary, the one-dimensional worm model presented here reproduces the main features observed in the development of wind ripples: increase in size through mergers, the appearance of size stabilization by slow growth, and a size approximately scaled by the sand grain diameter. Although not incorporating all of the complexities of wind ripple behavior, the model may capture the essence of their dynamics, a hypothesis that further simulations and experiments can test. Recent observations and computer simulations point to self-organization via mergers for other landforms as well [14,15]. The predicted implications of our worm model for landforms of this type are twofold: Growth and stabilization of characteristic scale can be explained by the same mechanism,

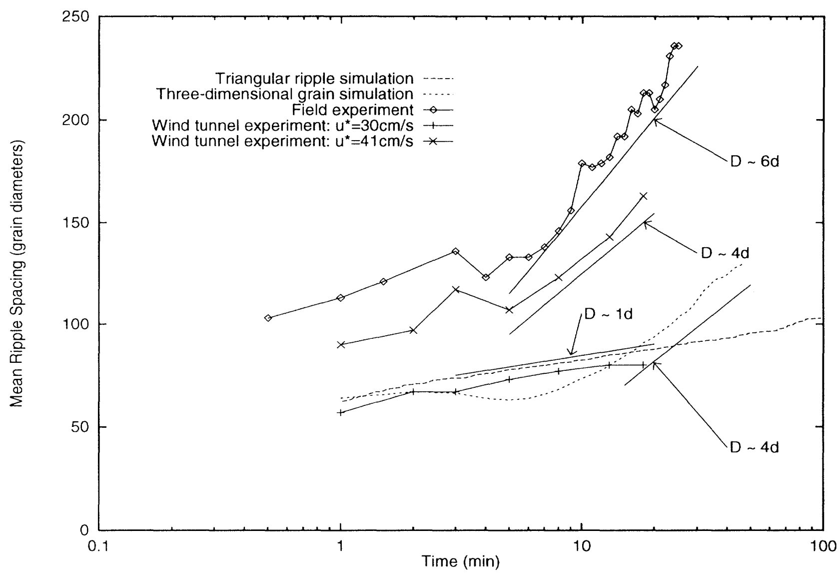

FIG. 4. Wind ripple spacing vs time. Spacing increase is consistent (within uncertainties) with *asymptotic* logarithmic dependence on time as predicted by the worm model. Field experiment conducted under variable wind conditions [2,5]; wind tunnel experiments performed with two values of the aerodynamical friction speed u^* [4]. Estimated uncertainties in ripple spacing: triangular ripple simulation, $< 3\%$; grain-by-grain simulations, $< 10\%$; field experiment, 10%–15%; wind tunnel experiments, uncertainties not provided. Inferred values of logarithmic slope of mean height vs time in terms of mean grain diameter d, assuming a ratio of spacing to ripple height of 10 for the physical measurements: triangle simulation, $D = 1d$; grain-by-grain simulation, $D = 4d$; field experiment, $D = 6d$; wind tunnel experiment, $u^* = 30$ cm/s, $D = 1d$; wind tunnel experiment, $u^* = 41$ cm/s, $D = 4d$.

and landform scale cannot be derived from a deterministic or a spatial continuum approach. The role of stochastic merging in other physical systems, such as underwater ripples and dunes and atomic-scale topographic patterns [16], remains to be investigated.

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