Collective Transport in Arrays of Small Metallic Dots

A. Alan Middleton and Ned S. Wingreen
NEC Research Institute, 4 Independence Way, Princeton, New Jersey 08540
(Received 18 March 1993)

Collective charge transport is studied in one- and two-dimensional arrays of small normal-metal dots separated by tunnel barriers. At temperatures well below the charging energy of a dot, disorder leads to a threshold for conduction which grows linearly with the size of the array. For short-ranged interactions, one of the correlation length exponents near threshold is found from a novel argument based on interface growth. The dynamical exponent for the current above threshold is also predicted analytically, and the requirements for its experimental observation are described.

PACS numbers: 73.40.Rw, 05.60.+w, 73.40.Gk

Systems exemplifying collective transport in quenched disorder include sliding charge-density waves (CDW's) [1,2], fluids in disordered media [3], and type-II superconductors [4]. For these dynamical systems, there does not yet exist a classification whereby the long-wavelength behavior can be predicted from the characteristics of the microscopic degrees of freedom. To study this question of universality experimentally requires systems where the microscopic degrees of freedom, the range of interactions, and the nature of the disorder are well understood. Here, we propose as a model system an array of small-capacitance normal-metal dots.

In this Letter we examine the low-temperature, nonlinear charge transport in such an array. The dots are treated as capacitively coupled conductors with charges allowed to tunnel between neighboring dots. In contrast with previous work [5], we explicitly include the effects of random offset charges on each dot and investigate the limit where the number of dots becomes large. We find that the onset of conduction occurs at a voltage V_T proportional to the linear array size. This sharp onset is to be contrasted with conduction in one- and two-dimensional disordered materials, where an increasing electric field leads to a smooth increase in conductivity [6,7]. One of the correlation lengths that diverges near this threshold is found from a general argument based on interface growth, while another is found by focusing on "slow points" which control the current. These correlation lengths determine the branching of current paths in the array and hence the current near onset. In particular, we predict that the current through linear and square arrays behaves as

$$I \sim (V/V_T - 1)^{\zeta}, \tag{1}$$

with $\zeta = 1, 5/3$ in dimensions $d = 1, 2$, respectively.

The array we study is depicted in Fig. 1. For a tunneling resistance R between dots large compared to the quantum resistance h/e^2, the state of the array is fully described by the number of electrons in each dot. The energy is then all electrostatic and is determined by a matrix of capacitances C_{ij}. We assume a constant capacitance C between neighboring dots and between the leads and adjacent dots, and a capacitance C_g between each dot and the back gate which underlies the entire array. The diagonal elements of C_{ij} are the sum of all capacitances associated with a dot and the off-diagonal elements are the negative of the interdot capacitances. The leads and back gate are taken to have infinite self-capacitance. We concentrate on the Coulomb-blockade regime, where the thermal energy is much smaller than the charging energies, i.e., $k_BT \ll e^2/[2\max(C, C_g)]$ (for 1 μm dots, this energy is ~ 1 meV [8]). We measure distances in units of the dot spacing.

Given the charge Q_i on each dot, the electrostatic energy is [5]

$$E = \frac{1}{2}\sum_{\text{dots } i,j}(Q_i + q_i)C_{ij}^{-1}(Q_j + q_j) + V_L Q_L + V_R Q_R + \sum_{\text{dots } i} V_i^{\text{ext}} Q_i, \tag{2}$$

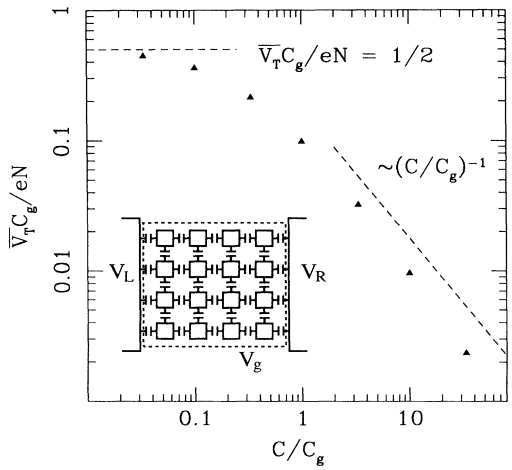

FIG. 1. The threshold voltage per dot, $\overline{V_T}/N$, in units of e/C_g, for conduction through a one-dimensional array of normal-metal dots as a function of C/C_g. The dashed lines show analytical predictions. A two-dimensional array of dots is shown in the inset. The indicated capacitance between dots is C, while the capacitance between each dot and the back gate (the dashed rectangle) is C_g. At $T = 0$, charges may only tunnel between neighboring dots if this lowers the total electrostatic energy. The voltages applied to the left lead, the right lead, and the gate are indicated as V_L, V_R, and V_G, respectively.

0031-9007/93/71(19)/3198(4)$06.00

where $Q_{L,R}$ are the charges on the leads, which are at voltages $V_{L,R}$. Disorder is included through the offset charges q_i which represent the effective charge on each dot due to nearby charged impurities. Large fluctuations in disorder will be compensated by an integral number of mobile charges, so that $0 \leq q_i < e$. The voltage V_i^{ext} on dot i due to the leads L, R and the gate g is $V_i^{\text{ext}} = \sum_x C_x \sum_j^x C_{ij}^{-1} V_x$, where $C_{L(R)} = C$ and $\sum_j^x$ is the sum over dots j which neighbor the electrode $x = L, R, g$. In general, the elements of the inverse capacitance matrix fall off exponentially with a screening length λ that increases with C/C_g [for $C \gg C_g$, $\lambda \approx (C/C_g)^{1/2}$].

At low temperatures, a charge may tunnel between dots only if such an event lowers the electrostatic energy of the array. The kinetic energy gained by the tunneling electron is assumed to be dissipated [9]. The tunneling rate from one configuration $S = (..., Q_i, ..., Q_j, ...)$ to another configuration $S' = (..., Q_i - 1, ..., Q_j + 1, ...)$, where i and j are neighboring dots or a dot and a neighboring lead, is given by

$$\nu_{S \to S'} = (e^2 R)^{-1} \theta(E(S) - E(S'))[E(S) - E(S')]. \quad (3)$$

This rate grows linearly with energy gain since the number of electrons available to tunnel is proportional to the relative shift of the Fermi surfaces in the dots. For arrays of a few junctions, numerical results on this model [5] compare well with experiment [8].

For large arrays, we find a second-order transition, with associated critical phenomena, which separates a static, nonconducting state from a dynamic, conducting state. The control parameter is the voltage difference between the leads. At low voltage differences, the array always relaxes to a static configuration, while at high voltage differences, charges traverse the array from one lead to the other.

An important question to ask is whether the conduction transition is hysteretic for a given realization of disorder. In one-dimensional systems, the current is a unique function of the applied voltages, regardless of the magnitude of λ [10]. In two-dimensional arrays at zero temperature, the current *can* depend on the history of the applied voltages. To within our numerical accuracy, however, the current is history independent in typical samples. It can furthermore be shown that the current is entirely independent of history in the limit of short screening lengths, $C/C_g \to 0$, to which we shall devote most of our attention.

We have numerically determined the dependence of the threshold voltage for conduction, $V_T(N) = V_L - V_R$, on the ratio C/C_g and on the linear system size N (for fixed gate voltage $V_g = 0$). We find that the threshold voltage is proportional to N,

$$\lim_{N \to \infty} \overline{V_T}(N) C_g / Ne = \alpha(C/C_g), \quad (4)$$

where the overbar represents an average over disorder.

The function $\alpha(C/C_g)$ for one-dimensional arrays is plotted in Fig 1. In the limit $C/C_g \to 0$, the voltage on a dot is just $(Q_i + q_i)/C_g$, as the capacitive coupling between dots is negligible. The schematic in Fig. 2(a) shows that in order to carry a current in this limit, the voltage difference across the array must be large enough to overcome $\approx N/2$ upward steps in the random potential. This observation gives $\alpha(C/C_g \to 0) = 1/2$. In the limit of large C/C_g (large λ), $\overline{V_T}$ can be estimated by balancing the "force" on the charges due to a charge-density gradient against the random potential gradient [11]. It is necessary to recognize that there is a stability limit for the dot-to-dot potential difference: At higher potential differences, charges will tunnel, reducing the potential across the tunneling barrier. Estimating the magnitude of the pinning forces to be given by this stability limit, we find [12] a maximum static density gradient of $\sim e/\lambda^2$; that is, a density change of $O(1)$ charge per screening length is allowed in regions separated by λ. This gives $\alpha \sim (C/C_g)^{-1}$ at large λ. Numerically, we find in $d = 1$ that $(C/C_g)\alpha \to 0.10(1)$, as $C/C_g \to \infty$.

We now discuss the approach to the conduction threshold in two dimensions, in order to elucidate the critical

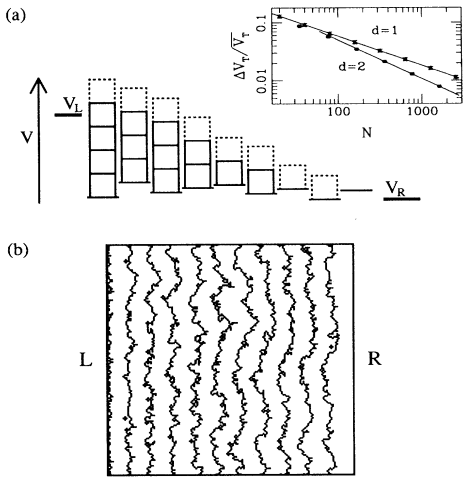

FIG. 2. (a) Schematic of dot voltages for a one-dimensional array below the threshold for conduction, in the limit of short screening length λ. Each square indicates an increase in the on-site voltage by e/C_g due to an added charge; the relative offset in the voltages is caused by quenched disorder. If the left-lead voltage V_L is further raised by e/C_g, charges will tunnel onto the array until stopped by the next upward step, as indicated by the dashed squares. The inset shows the calculated sample-to-sample fluctuation of threshold voltages, V_T, in one and two dimensions as a function of linear system size, N, with fits described in the text [Eq. (6)]. (b) Contours of constant charge occupation in a 160^2 array at the threshold V_T (contour spacing is 5 charges). Successive contours coincide with the distance to which charge flows for various voltages below threshold, $V_L < V_T$. Conduction occurs at the voltage where the charge first reaches the right lead.

behavior of the correlation lengths and current. We concentrate on the limit $C/C_g \to 0$, that is, λ small, in order to maximize the number of effective degrees of freedom and to eliminate hysteresis. Taking V_R to be fixed and raising V_L, charge moves from the left lead onto the array. The condition for a charge to overcome the Coulomb barrier and tunnel from site i to neighboring site j is

$$V_i > V_j + e/C_g. \tag{5}$$

At a given V_L, this advance of charge is halted when $V_i \leq V_j + e/C_g$ everywhere. Though the tunneling is stochastic, the static configuration at any $V_L < V_T$ is entirely determined by the disorder realization. The distance to which charge penetrates therefore defines a unique interface (given $Q_i = 0$ initially). When V_L is raised by e/C_g, charge is added to each point on the interface and *therefore the interface must advance by at least one lattice spacing.* In addition, the interface may advance further at some points if the local disorder is favorable. The motion of this interface is depicted in Fig. 2(b); conduction occurs when the interface reaches the right lead. Numerical calculations of this threshold give $\alpha(C/C_g \to 0) = 0.338(1)$ in $d = 2$.

The dynamics of the Coulomb-blockade condition Eq. (5) make the interface "motion" with increasing V_L similar to the stochastic growth of interfaces in models without quenched spatial disorder, such as the one due to Eden [13]. The results on the Kardar-Parisi-Zhang (KPZ) equation for a $(d-1)$-dimensional interface [14] subject to short-range correlated noise are therefore useful in understanding the behavior of a d-dimensional array of dots. The KPZ equation will describe the long-wavelength behavior even when $C/C_g \neq 0$, as the interactions are short range, rotation is a symmetry on large scales, and the speed of interface advance lacks large fluctuations. This is to be contrasted with the usual motion of interfaces at small velocities through random media, where the interface can be pinned for some time at one point, resulting in long time correlations and exponents distinct from those of the KPZ equation [15].

In the case $d = 2$, the results for KPZ interfaces imply that the width of the interface must scale as $V_L^{1/3}$ [14]. Furthermore, the fluctuations in the position of maximum advance of the interface behave as $\sim V_L^{1/3}(\ln V_L)^{1/2}$. The rms fluctuations ΔV_T in V_T and the mean threshold voltages $\overline{V_T}$ therefore behave as

$$\Delta V_T/\overline{V_T} \sim N^{-2/3}(\ln N)^{1/2}, \tag{6}$$

$$\overline{V_T}(N)C_g/eN - \alpha \sim N^{-2/3}(\ln N)^{1/2}, \tag{7}$$

for an $N \times N$ array [16] (for $d = 1$, $\Delta V_T/V_T \sim N^{-1/2}$). The fluctuations in the threshold voltage as a function of size may be used to define a finite-size scaling exponent ν_T via $N \sim (\Delta V_T/\overline{V_T})^{-\nu_T}$. This length, besides giving the fluctuations in V_T, determines the finite-size crossover in quantities such as the polarization of the array [2]. From Eq. (6), we find $\nu_T = 3/2$. Numerical simulations on systems up to size $N = 2560$ are fit very well by Eq. (6), as shown in the inset in Fig. 2(a), and by Eq. (7).

The current in the one-dimensional model with only on-site interactions can be understood in detail. For voltages much greater than threshold, $v \equiv (V_L - V_T)/V_T \gg 1$, the charge gradient across each junction is much greater than 1. By Eq. (3), the current is then approximately

$$I \approx (e/2RC_g)v. \tag{8}$$

In contrast, near threshold, the discreteness of the charges and the disorder become important. The excess charge gradient above the threshold configuration is composed of steps that occur at well separated "slow points," located where the potential drop between dots in the threshold configuration is small compared to e/C_g. It can be shown [12] that the current is given by the fastest slow point; the tunneling rate across this point is, on average, $(V - \overline{V_T})/eRN$. Interestingly, near threshold this also gives Eq. (8). We therefore find $\zeta = 1$ for Eq. (1) in $d = 1$. As shown in Fig. 3(a), Eq. (8) is consistent with our numerical results near and far from threshold.

The pattern of current flow in a typical two-dimensional array is shown in Fig. 3(b). At voltages just above threshold, $V_L - V_T \ll e/C_g$, the current is in general carried on a single path, with little or no branching.

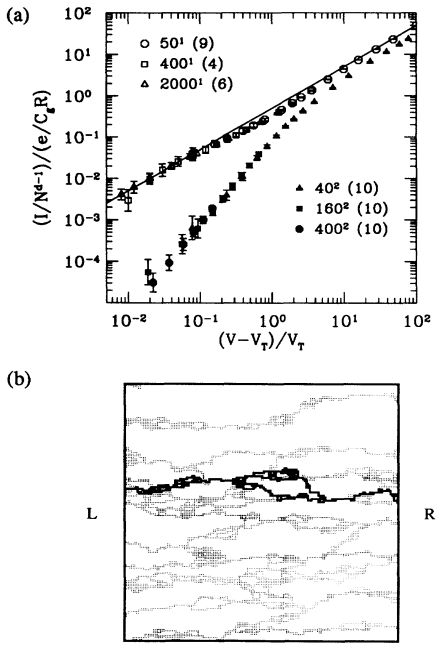

FIG. 3. (a) Plot of current-voltage relationship near threshold for one- and two-dimensional arrays of various sizes. The numbers in parentheses give the number of disorder realizations. For one-dimensional arrays, the current both near and far from threshold is well fit by Eq. (8). The data for the $d = 2$ arrays are approximately fit by $I \sim (V - V_T)^{2.0}$ at the lowest currents shown, though the local slope on the log-log plot has not converged. (b) Current paths in a two-dimensional array (of size 160^2) at two voltages near threshold. Very near threshold, the current flows in a single narrow channel (dark line). Multiple branching channels are shown for a voltage $3e/C_g$ above threshold (light lines).

This path is exactly the one with the minimal number of upward steps in the potential between the two leads. Previous work [14] shows that such paths have transverse fluctuations $\sim n^{2/3}$, where n is the distance from the left lead; this is consistent with our numerical results. Increasing the voltage to a few times e/C_g above threshold opens multiple channels which branch and reconnect, as shown in Fig. 3(b). Note that at voltages V_L exceeding threshold by $O(eN^{1/3}(\ln N)^{1/2}/C_g)$, current can in principle flow anywhere in the array, as the charge invasion interface contacts the right lead at each point. However, near threshold, all but a small fraction of the current is confined to a few major current-carrying paths. The selection of these paths out of all possible paths results from a characteristic length for branching, as we now discuss.

Any current-carrying channel between the two leads must have $(V_L - V_T)C_g$ excess steps in the charge density relative to the threshold configuration. This gives a correlation length $\xi_\parallel = eN/(V_L - V_T)C_g$ which separates the steps in the excess charge, as in $d = 1$. This length determines the separation between branch points along the channels [12]. The channels therefore wander transversely a distance $\xi_\parallel^{2/3}$ between branch points, giving a channel separation of $\xi_\perp \sim v^{-2/3}$. The current through each channel behaves as $\sim (e/2RC_g)v$, since each segment between branch points is one dimensional. The current through the array is then given by

$$I \sim (e/2RC_g)vN/\xi_\perp \sim (e/2RC_g)Nv^{5/3}, \qquad (9)$$

resulting in $\zeta(d = 2) = 5/3$.

Our numerical results for the transport in two-dimensional systems are shown in Fig. 3(b). The current-voltage relationship is approximately fit by $I/N \sim v^{2.0\pm0.2}$, for v near 10^{-1}, but the slope on a log-log plot does not converge in the range we have studied numerically. To observe the true exponent requires arrays larger than 400^2, either numerical or experimental.

In conclusion, we have determined the threshold for conduction in arrays of small normal-metal dots with disorder. By examining correlation lengths that describe the separation of parallel current paths and the distance between dynamically important "slow points" we have determined the transport behavior near the threshold. The critical exponents for the current and correlation lengths which we have derived using the KPZ interface model are distinct from those found for elastic media [1] and fluid flow [3,17]. These differences are clearly related to the novel features of this system, namely, the discreteness of the carriers and (quantum) stochastic flow, which result in (a) an always advancing charge interface below threshold and (b) the nonlocal selection of current paths above threshold.

We wish to thank Paul McEuen for encouraging our interest in arrays, and Chao Tang for many valuable discussions.

[1] For reviews, see *Charge Density Waves in Solids*, edited by L. P. Gorkov and G. Grüner (Elsevier, Amsterdam, 1989).

[2] O. Narayan and D. S. Fisher, Phys. Rev. Lett. **68**, 3615 (1992); Phys. Rev. B **46**, 11 520 (1992); A. A. Middleton and D. S. Fisher, Phys. Rev. B **47**, 3530 (1993).

[3] N. Martys, M. Cieplak, and M. O. Robbins, Phys. Rev. Lett. **66**, 1058 (1991).

[4] D. S. Fisher, in *Nonlinearity in Condensed Matter*, edited by A. R. Bishop *et al.* (Springer-Verlag, New York, 1987).

[5] U. Geigenmüller and G. Schön, Europhys. Lett. **10**, 765 (1989); N. S. Bakhvalov *et al.*, Zh. Eksp. Teor. Fiz. **95**, 1010 (1989) [Sov. Phys. JETP **68**, 581 (1989)].

[6] F. Tremblay *et al.*, Phys. Rev. B **40**, 3387 (1989); D. Shahar and Z. Ovadyahu, Phys. Rev. Lett. **64**, 2293 (1990).

[7] Also in constrast with the hopping regime, the disorder in our model does not lead to a broad range of hopping rates and hence sample-to-sample fluctuations are less evident.

[8] P. Delsing, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992).

[9] For small-capacitance and high-resistance junctions, the effects of small finite temperatures and of cotunneling, in which electrons tunnel coherently through two barriers, are minimal [D. V. Averin and Yu. V. Nazarov, in *Single Charge Tunneling* (Ref. [8])].

[10] A "no-passing" rule for the integrated current through the junctions can be used to show that, at fixed voltage, each initial state has the same average velocity at long times; see A. A. Middleton, Phys. Rev. Lett. **68**, 670 (1992).

[11] C. P. Bean, Phys. Rev. Lett. **8**, 250 (1962).

[12] A. A. Middleton and N. S. Wingreen (to be published).

[13] M. Eden, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, edited by F. Neyman (University of California Press, Berkeley, 1961), Vol. IV.

[14] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985); S. Roux, A. Hansen, and E. L. Hinrichsen, J. Phys. A **24**, L295 (1991); L.-H. Tang, B. M. Forrest, and D. E. Wolf, Phys. Rev. B **45**, 7162 (1992).

[15] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992); see also Ref. [3] and references therein.

[16] We cite results here for a square $N \times N$ array. For an anisotropic array, with a linear distance between the leads of M and width N, the fluctuations in the threshold voltage scale as $N^{1/2}$, for $MN^{-2/3} \ll 1$. This method can also be applied to CDW models: Numerically, the interface width is constant, reflecting elastic behavior, but the average velocity has $t^{-d/2}$ fluctuations, leading to the known value $\nu_T = 2/d$ (Ref. [2]).

[17] H. Takayasu, Phys. Rev. Lett. **63**, 2563 (1989); O. Narayan and D. S. Fisher (to be published).