Gürses's Type (*b*) Transformations Are Neighborhood Isometries

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Following an idea close to one given by C. G. Torre (private communication), we prove that Riemannian spaces (M,g) and (M,h) that are related by a Gürses's type (b) transformation [M. Gürses, Phys. Rev. Lett. **70**, 367 (1993)] or, equivalently, by a Torre-Anderson generalized diffeomorphism [C. G. Torre and I. M. Anderson, Phys. Rev. Lett. **70**, 3525 (1993)] are *neighborhood isometric*, i.e., every point $\mathbf{x}$ in M has a corresponding diffeomorphism ϕ of a neighborhood V of $\mathbf{x}$ onto a generally different neighborhood W of $\mathbf{x}$ such that $\phi^*(H|_W)=g|_V$.

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I. Introduction.—There have been two divergent opinions concerning those generalized symmetries of the Einstein vacuum field equations which are found by Gürses [1] and which he designated as *type (b)*. One opinion, which both Gürses and the authors once entertained, was that some type (b) transformations can be used (at least in principle) to generate new exact solutions. An opinion which suggests the opposite was given by Torre and Anderson [2] in their analysis of generalized symmetries. They stated that the *generalized diffeomorphism symmetries* which they discovered are physically trivial, and we shall see [3] that a type (b) generator is equal to a generalized diffeomorphism generator apart from an inconsequential Lorentz transformation term. The reason for their opinion has been explained to us by Torre [4] and it is our efforts to place his explanation on secure mathematical ground that has led to this paper.

The objectives of this paper are to introduce the concept of *neighborhood-isometric [5] spacetimes* (an explication of the description furnished to us by Torre) and to prove that spacetimes which are related by a type (b) transformation are neighborhood isometric. Before we do that, we shall define type (b) transformations in a way which will facilitate our proof. The vacuum condition will be ignored in our definition because, as we shall further stress in the discussions of Sec. VII, this condition is not required for the proof of our theorem. In Sec. VII, we shall also explain our own viewpoint on whether spacetimes which are related by a type (b) transformation are physically indistinguishable. The theorem which we shall prove implies that no type (b) transformation can be used to generate a new exact analytic solution. On the other hand, we shall state a theorem which remains to be proved before one can assert correctly and without reservation that spacetimes which are related by a type (b) transformation are physically equivalent.

II. Definition of the type (b) *transformations.*—Our definition of the type (b) one-parameter family of transformation differs from but is equivalent to that of Gürses [1]. The equivalence and the correspondences with his notations will be detailed in a lengthier paper [6] which will cover all of his transformation types.

The parameter will be denoted by ϵ (Gürses uses "ϵ_0"). At $\epsilon=0$, suppose one prescribes the following objects:

(1) A spacetime $(M,g(0))$. (2) An orthonormal tetrad of 1-forms $e^b(0)$ $(b=1,2,3,4)$ with the domain M. It is granted that M is restricted so that a tetrad exists. The manifold and all prescribed functions are assumed to be $\mathbf{C}^\infty$. (3) A vector field $\mathbf{a}(0)$ whose domain is M. Then the type (b) transformation

$$(M,g(0),\{e^b(0)\},\mathbf{a}(0))\rightarrow(M,g(\epsilon),\{e^b(\epsilon)\},\mathbf{a}(\epsilon))$$

is defined as follows.

Definition: First define the real-value fields a^b by the equation

$$\mathbf{a}(0)=a^b\mathbf{e}_b(0)\,, \tag{1}$$

where $\{\mathbf{e}_b(0)\}$ is the dual basis of $\{e^b(0)\}$. (All p-vectors are denoted by boldface letters, and all p-forms by lightface.) Let $e^b(\epsilon)$ denote the integrals over a maximally extended connected interval $J\subset R^1$ of the family of equations

$$\partial e^b(\epsilon)/\partial\epsilon=-da^b-a^c\Gamma_{cd}(\epsilon)\eta^{db}\,, \tag{2}$$

where the connection 1-forms $\Gamma_{cd}(\epsilon)$ are defined in terms of $e^b(\epsilon)$ by the familar structural equations

$$d\wedge e^b(\epsilon)=e^c(\epsilon)\wedge\Gamma_{cd}(\epsilon)\eta^{db},\quad \Gamma_{cd}(\epsilon)=-\Gamma_{dc}(\epsilon)\,, \tag{3}$$

and $\eta^{db}:=0$ if $d\neq b$ and $\eta^{11}=\eta^{22}=\eta^{33}=-\eta^{44}:=1$. Also, let

$$g(\epsilon):=\eta_{ab}e^a(\epsilon)\otimes e^b(\epsilon)\,, \tag{4}$$

where $\eta^{ab}\eta_{bc}=\delta^a_c$. Thereupon, $(M,g(\epsilon))$ is automatically a spacetime for which $\{e^b(\epsilon)\}$ is an orthonormal tetrad. Upon letting $\{\mathbf{e}_b(\epsilon)\}$ be the dual basis of $\{e^b(\epsilon)\}$ and

$$\mathbf{a}(\epsilon):=a^b\mathbf{e}_b(\epsilon)\,, \tag{5}$$

it is easily shown that Eq. (2) is expressible as

$$\partial e^b(\epsilon)/\partial\epsilon=-e^c(\epsilon)[\nabla_c a^b]\,, \tag{6}$$

where $\nabla_c a^b$ are the orthonormal components of the covariant derivative of $\mathbf{a}(\epsilon)$ on the spacetime $(M,g(\epsilon))$.

That completes our definition. To guarantee existence, one may assume that M is compact (which includes the

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possibility that it is a compact subspace of another given manifold). In any case, we shall grant that the solution of Eqs. (2) and (3) over a nontrivial interval J exists and is $\mathbf{C}^\infty$. We point out that the above definition and all conclusions of this paper are also applicable if the prescribed orthonormal components a^b are suitably chosen functions of ϵ as well as of the spacetime points. We shall (following Gürses) include this generalization in the type (*b*) category.

III. Neighborhood-isometric spacetimes.—Definition: For any set X, let 1_X denote the function whose domain is X such that $1_X(x)=x$ for all x in X. In other words 1_X is the identity map on X. For any function f, note that $f\circ 1_X$ is the restriction of f to X.

Definition: Let (M,g) and (M,h) be any spacetimes and $\mathbf{x}$ be any point in M. Suppose there exist neighborhoods V,W of $\mathbf{x}$ and a diffeomorphism ϕ of V onto W such that $\phi^*(h\circ 1_W)=g\circ 1_V$. Then we shall say that (M,g) and (M,h) are *neighborhood isometric at* $\mathbf{x}$ and write this as

$$(M,g) < \mathrm{ni},\mathbf{x} > (M,h)\,.$$

The proof of the following theorem is a pleasant exercise.

Theorem: For given $\mathbf{x}\in M$, $< \mathrm{ni},\mathbf{x} >$ is an equivalence relation.

Definition: We shall say that (M,g) and (M,h) are neighborhood isometric and write

$$(M,g) < \mathrm{ni},M > (M,h)$$

if $(M,g) < \mathrm{ni},\mathbf{x} > (M,h)$ for all $\mathbf{x}$ in M.

It is clear that $< \mathrm{ni},M >$ is an equivalence relation. The key theorem of this paper can now be simply expressed as follows in terms of the notations employed in our definition of a type (*b*) transformation.

Theorem: For all ϵ in J,

$$(M,g(\epsilon)) < \mathrm{ni},M > (M,g(0))\,. \tag{7}$$

The proof of the above theorem will be simple once we have proved a lemma which will be formulated in the next section. This lemma involves an explicit construction of the diffeomorphism.

IV. Formulation of a lemma.—Definitions: For any given point $(\epsilon_0,\mathbf{x}_0)$ in $J\times M$, let $j_0\subset J$ be any connected open neighborhood of ϵ_0 (in the topology of J relative to R^1) and let $U\subset M$ be any neighborhood of $\mathbf{x}_0$ such that there exists a chart σ whose domain is U. Also, let X be the range of σ and $x:=(x^1,x^2,x^3,x^4)=\sigma(\mathbf{x})$ for all $\mathbf{x}$ in U.

Definitions: Restricting j_0 and U if necessary to guarantee existence, and letting ϵ be any point in j_0, we define $f(\epsilon)$ to be that one-one function whose domain is X, whose range is

$$Y(\epsilon):=[\text{range of } f(\epsilon)]\subset R^4\,, \tag{8}$$

and which satisfies the familiar flow equation ($\beta=1,2,3,4$)

$$\frac{\partial f(\epsilon,x)}{\partial\epsilon}+a^\beta(\epsilon,x)\frac{\partial f(\epsilon,x)}{\partial x^\beta}=0\,, \tag{9}$$

and the initial condition

$$f(\epsilon_0)=1_X:=\text{identity map on } X\,, \tag{10}$$

where $f(\epsilon,x):=f(\epsilon)(x)$ and where $a^\beta(\epsilon,x)$ are the components of $\mathbf{a}(\epsilon,\mathbf{x}):=\mathbf{a}(\epsilon)(\mathbf{x})$ relative to the chart σ. Let y be any point in $Y(\epsilon)$ and

$$h(\epsilon):=[f(\epsilon)]^{-1},\quad h(\epsilon,y):=h(\epsilon)(y)\,, \tag{11}$$

whereupon it is easily shown that Eq. (9) is equivalent to the equation ($\mu=1,2,3,4$)

$$\partial h^\mu(\epsilon,y)/\partial\epsilon=a^\mu(\epsilon,h(\epsilon,y))\,. \tag{12}$$

From Eqs. (8) and (10), $Y(\epsilon_0)=X$. So we can and we do further restrict j_0, if necessary, so that there exists an open set $\mathcal{Y}\subset R^4$ which satisfies

$$\sigma(\mathbf{x}_0)\in\mathcal{Y}\subset Y(\epsilon) \text{ for all } \epsilon \text{ in } j_0\,. \tag{13}$$

Then the sets defined below are not empty.

Definitions: For all ϵ in j_0, let

$$\tau:=(1_{\mathcal{Y}})\circ\sigma,\quad \mu(\epsilon):=1_{\mathcal{Y}}\circ f(\epsilon)\circ\sigma\,, \tag{14}$$

which are clearly charts that both have the range $\mathcal{Y}$ and that have the domains

$$V:=\{\mathbf{x}\in U|\sigma(\mathbf{x})\in\mathcal{Y}\} \tag{15}$$

and

$$W(\epsilon):=\{\mathbf{x}\in U{:}f(\epsilon,\sigma(\mathbf{x}))\in\mathcal{Y}\}\,, \tag{16}$$

respectively. For all ϵ in j_0, let

$$\phi(\epsilon):=[\mu(\epsilon)]^{-1}\circ\tau\,, \tag{17}$$

which maps V onto $W(\epsilon)$.

Lemma: The diffeomorphism $\phi(\epsilon)$ satisfies

$$[\phi(\epsilon)]^*[g(\epsilon)\circ 1_{W(\epsilon)}]=g(\epsilon_0)\circ 1_V \text{ if } \epsilon \text{ is in } j_0\,. \tag{18}$$

Note that V and $W(\epsilon)$ are both neighborhoods of $\mathbf{x}_0$. So the above lemma implies that, for all ϵ in j_0, $(M,g(\epsilon))$ and $(M,g(\epsilon_0))$ are neighborhood isometric at $\mathbf{x}_0$.

V. Proof of the lemma.—An alternative form of Eq. (6) is as follows:

$$\frac{\partial e^b(\epsilon)}{\partial\epsilon}=-\mathcal{L}_{\mathbf{a}(\epsilon)}e^b(\epsilon)+[\mathbf{a}(\epsilon)\Gamma_{cd}(\epsilon)]e^d(\epsilon)\eta^{cb}\,, \tag{19}$$

where $\mathbf{a}(\epsilon)\Gamma_{cd}(\epsilon)$ is the value of the linear functional $\Gamma_{cd}(\epsilon)$ corresponding to the vector $\mathbf{a}(\epsilon)$. The second term on the right side of Eq. (19) represents (when multiplied by $\delta\epsilon$) an infinitesimal Lorentz transformation of $\{e^b(\epsilon)\}$ and cannot, therefore, contribute to the variation of the metric tensor with respect to ϵ. In fact, a brief calculation employing Eq. (4) yields

$$\frac{\partial g(\epsilon)}{\partial\epsilon}=-\mathcal{L}_{\mathbf{a}(\epsilon)}g(\epsilon)\,. \tag{20}$$

The theorem of this paper, and even a much stronger form of the theorem, would now be self-evident if $\mathbf{a}(\epsilon)$ were independent of ϵ. However, the fact that $\mathbf{a}(\epsilon)$ does not generally depend on ϵ and is determined in an intricate way by the prescribed functions at $\epsilon=0$ has led to some doubts as indicated by our opening remarks. Therefore, the pedestrian proof which we are using in this paper may be appreciated by at least some readers. We continue by defining a family of real-valued functions $g_{\mu\nu}(\epsilon)$ and $\gamma_{\alpha\beta}(\epsilon)$.

Definitions: Let $g_{\mu\nu}(\epsilon)$ have the domain X and be the components of $g(\epsilon)$ relative to the charge σ. Let $\gamma_{\alpha\beta}(\epsilon)$ have the domain $Y(\epsilon)$ and be the components of $g(\epsilon)$ relative to the chart $f(\epsilon)\circ\sigma$. Let

$$g_{\mu\nu}(\epsilon,x):=g_{\mu\nu}(\epsilon)(x),\quad \gamma_{\alpha\beta}(\epsilon,y):=\gamma_{\alpha\beta}(\epsilon)(y)\,.$$

Then, at each y in $Y(\epsilon)$,

$$\gamma_{\alpha\beta}(\epsilon,y)=g_{\mu\nu}(\epsilon,h(\epsilon,y))\frac{\partial h^\mu(\epsilon,y)}{\partial y^\alpha}\frac{\partial h^\nu(\epsilon,y)}{\partial y^\beta}\,. \tag{21}$$

Furthermore, Eq. (20) becomes relative to σ:

$$\frac{\partial g_{\mu\nu}(\epsilon,x)}{\partial\epsilon}+a^\alpha(\epsilon,x)\frac{\partial g_{\mu\nu}(\epsilon,x)}{\partial x^\alpha}+\frac{\partial a^\alpha(\epsilon,x)}{\partial x^\mu}g_{\alpha\nu}(\epsilon,x)+\frac{\partial a^\alpha(\epsilon,x)}{\partial x^\nu}g_{\alpha\mu}(\epsilon,x)=0\,. \tag{22}$$

Equations (11), (12), (21), and (22), and the chain rule, now imply

$$\partial\gamma_{\alpha\beta}(\epsilon,y)/\partial\epsilon=0\,. \tag{23}$$

So, upon setting $\epsilon=\epsilon_0$ in the left side of Eq. (21) and taking into account the facts that the domains of $\gamma_{\alpha\beta}(\epsilon)$ and $\gamma_{\alpha\beta}(\epsilon_0)=g_{\alpha\beta}(\epsilon_0)$ are $Y(\epsilon)$ and $Y(\epsilon_0)=X$, respectively, we obtain from Eqs. (13) and (23)

$$g_{\alpha\beta}(\epsilon_0,y)=g_{\mu\nu}(\epsilon,h(\epsilon,y))\frac{\partial h^\mu(\epsilon,y)}{\partial y^\alpha}\frac{\partial h^\nu(\epsilon,y)}{\partial y^\beta}\quad\text{for all } \epsilon \text{ in } j_0 \text{ and } y \text{ in } \mathcal{Y}\,. \tag{24}$$

Employing the definitions (14), (15), and (16), we see that Eq. (24) is the component form of the pullback equality

$$(\tau^{-1})^*[g(\epsilon_0)\circ 1_V]=[\mu(\epsilon)^{-1}]^*[g(\epsilon)\circ 1_{W(\epsilon)}]\,. \tag{25}$$

Equation (18) then follows from Eqs. (17) and (25). That completes the proof of the lemma.

VI. Proof of the theorem.—We merely sketch the proof since the reader can fill in the details without difficulty. Consider any given number ϵ in J and let $|0,\epsilon|$ be the closed interval with end points $0,\epsilon$. Let $\mathbf{x}$ be any point in M. From the lemma, every number ϵ_0 in J can be covered by at least one open interval j_0 such that

$$(M,g(\epsilon'))<\mathrm{ni},\mathbf{x}>(M,g(\epsilon_0))$$

for all ϵ' in j_0. The Heine-Borel covering theorem then implies that there exists a finite sequence of numbers $\epsilon_1,\ldots,\epsilon_A,\ldots,\epsilon_N$ and open subintervals $j_1,\ldots,j_A,\ldots,j_N$ of J such that ϵ_A lies on $|0,\epsilon|$, j_A covers ϵ_A, j_A and j_{A+1} overlap for $A=1,\ldots,N-1$, the union of the intervals j_A covers $|0,\epsilon|$ and

$$(M,g(\epsilon'))<\mathrm{ni},\mathbf{x}>(M,g(\epsilon_A))$$

for all ϵ' in j_A. Therefore, since $<\mathrm{ni},\mathbf{x}>$ is an equivalence relation, we infer that $(M,g(\epsilon))$ and $(M,g(0))$ are neighborhood isometric at $\mathbf{x}$. However, ϵ and $\mathbf{x}$ were arbitrarily chosen members of J and M, respectively, so the theorem [Eq. (7)] follows.

VII. Discussion.—In retrospect, the premise that $(M,g(\epsilon))$ is a vacuum spacetime is never used in the proof and is superfluous for this paper. Premises respecting the matter tensor are irrelevant both for the definition of the type (b) transformation and for the validity of the theorem. This fact strongly supports but does not prove the opinion of Torre and Anderson that their generalized diffeomorphisms are physically trivial [2], or, as we prefer to express it, that spacetimes which are related to a type (b) transformation are physically indistinguishable. We shall now review this opinion.

It is certain from the theorem proved in this paper that every point of the manifold M can be covered by one-parameter families of neighborhoods $V(\epsilon)$ and $W(\epsilon)$ such that the spacetimes $(V(\epsilon),g(0)|_{V(\epsilon)})$ and $(W(\epsilon), g(\epsilon)|_{W(\epsilon)})$ are physically equivalent for all ϵ. This dashes any hope that type (b) transformations can be used to generate new exact analytic solutions of the Einstein field equations *regardless of the premises made concerning the matter tensor.* However, we hesitate to go beyond the foregoing statements.

We propose that any given spacetimes (M,g) and (N,h) are *physically indistinguishable* if and only if they are isometric [5] (and not just neighborhood isometric) or have extensions which are isometric. (These extensions need not be maximal.) Therefore, to prove that $(M,g(\epsilon))$ and $(M,g(0))$ are physically indistinguishable *on more than just a local level,* one must prove that they are isometric or have isometric extensions. Initial efforts in that direction indicate that the task of proving this or of finding a counterexample may not be trivial. Nor do we have any reason to believe that the conjectured theorem is true. There are three related research paths which may be helpful in resolving the issue: (1) One can focus attention on analytic spacetimes and analytic type (b) transformations. (2) One can test the validity of the conjectured theorem for two-dimensional Lorentzian manifolds. (3) One can investigate the effects on type (b) transformations of critical points of $\mathbf{a}(0)$. We leave the above ventures for interested readers.

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[1] M. Gürses, Phys. Rev. Lett. **70**, 367 (1993).

[2] C. Torre and I. Anderson, Phys. Rev. Lett. **70**, 3525 (1993).

[3] This can be seen by inserting Eq. (16) of our paper into Eq. (14) and then comparing the result with Eq. (6) of Ref. [2].

[4] C. G. Torre (private communication).

[5] This concept is based on the concept of isometric Riemannian or semi-Riemannian manifolds as defined by R. K. Sachs and H. Wu, *General Relativity for Mathematicians* (Springer-Verlag, New York, 1977), p. 4. The concept of neighborhood isometry is our own invention, to the best of our knowledge. It is not to be confused with the concept of local isometry as defined by Sachs and Wu on p. 4 of their book.

[6] F. J. Ernst and I. Hauser (to be published).