Behavior of Solitons in Random-Function Solutions of the Periodic Korteweg-de Vries Equation

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(Received 28 July 1993)

I develop a general approach for computing *random-function solutions* of the periodic Korteweg-de Vries (KdV) equation using the inverse scattering transform (IST) in the hyperelliptic function representation. I exploit IST to construct *realizations* of KdV random processes which have *power-law spectra*, $k^{-\gamma}$ (k the wave number, γ a constant), and uniformly distributed *random IST phases* on $(-\pi,\pi)$. IST characterizes these realizations in terms of solitons moving in a sea of background radiation and is thus able to extract solitons, by *nonlinear filtering* techniques, from *complex, random motions* described by the KdV equation.

PACS numbers: 47.27.Te, 47.10.+g

Measurements of nonlinear fluid dynamical systems often reveal power spectra which have an approximate power law ($k^{-\gamma}$) over a wide wave number or frequency range. The number of physical examples is too numerous to list, but includes three-dimensional turbulence, geophysical fluid dynamical turbulence, and internal and surface wave motions (see, for example, [1,2] and cited references). Results of this type have helped motivate the study of *random functions* from the perspective of Fourier and power spectral analysis [3,4], in which one often writes the space-time evolution of a dynamical random process as

$$u(x,t)=\sum_{n=1}^{\infty} C_n \cos(k_n x-\omega_n t-\phi_n) . \qquad (1)$$

$C_n=\sqrt{2P(k_n)\Delta k}$, $P(k_n)$ is the power spectrum, $\omega_n(k_n)$ is the dispersion relation, and the ϕ_n are uniformly distributed random phases on the interval $(-\pi,\pi)$; each set of phases generates a distinct realization of the random process (1). Solutions to (1) for power-law spectra, often referred to as *colored random noises*, have a fractal dimension $D=2/(\gamma-1)$, $1\le \alpha \le 3$ [5].

In the present paper I consider the possibility of extending (1) to a particular nonlinear problem, the Korteweg-de Vries (KdV) equation, which is completely integrable by the inverse scattering transform (IST) in (1+1) dimensions [6,7]. This study is motivated by the fact that IST is a generalization of Fourier analysis to nonlinear problems [8]. Thus one has the full spectral structure of IST to aid in understanding nonlinear interactions. In what sense does a nonlinear generalization of (1) exist for KdV? Can KdV tell us something about the behavior of solitons in systems with power-law spectra? Can solitons be systematically studied even though for most of their lifetimes they may not be visible because of their high spatial density, strong nonlinear interactions, and/or the presence of background radiation? To address these and other questions, I consider here a scenario by which the KdV equation may be viewed as a nonlinear random process. Random-function solutions to KdV are defined by means of the mathematical machinery of IST, which contains several parameters (the IST phases) that are here viewed as random variables. I give a preliminary overview of KdV random processes, discuss some of their properties, and present a numerical example.

This paper may be viewed as a natural extension of the fundamental work of Zabusky and Kruskal (ZK) [6] who used the sine-wave initial conditions to discover the soliton, i.e., a delta-function spectrum at $t=0$. Here I consider the case of random initial conditions with a power-law spectrum at $t=0$.

KdV describes small-but-finite-amplitude, long-wave motion [9,10]:

$$\eta_t+c_0\eta_x+\alpha\eta\eta_x+\beta\eta_{xxx}=0 . \qquad (2)$$

Equation (2) governs the space-time evolution of the nonlinear field, $\eta(x,t)$, here assumed to be spatially periodic, $\eta(x+L,t)=\eta(x,t)$, for $0\le x\le L$, L the period. The coefficients of (2) are constant and have values that depend upon the physical application; these include surface waves, internal waves, Rossby waves, plasma waves, equatorial motions, and bores [2,8].

The general spectral solution to periodic KdV (2) [11-13] is written as a *linear superposition of nonlinearly interacting, nonlinear waves (hyperelliptic functions)*, $\mu_j(x,t)$:

$$\lambda\eta(x,t)=-E_1+\sum_{j=1}^{N}[2\mu_j(x,t)-E_{2j}-E_{2j+1}] . \qquad (3)$$

$\lambda=\alpha/6\beta$ and the E_i ($1\le i\le 2N+1$) are constant eigenvalues derived from Floquet theory for the time-independent Schrödinger equation. Since (3) reduces to a linear Fourier series (1) in the limit of small amplitude motion [14] (for particular deterministic values of the amplitudes and phases in terms of scatterning transform variables), I refer to (3) as *a nonlinear Fourier series*. The μ_j evolve in space according to the following system of coupled, nonlinear, ordinary differential equations (ODEs):

$$\frac{d\mu_j}{dx}=2i\sigma_j R^{1/2}(\mu_j)\Big/\prod_{\substack{k=1\\ j\neq k}}^{N}(\mu_j-\mu_k) , \qquad (4)$$

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where $1 \leq j \leq N$ and

$$R(\mu_j) = \prod_{k=1}^{2N+1} (\mu_j - E_k) . \quad (5)$$

The $\sigma_j = \pm 1$ are the signs of the square root of the function $R(\mu_j)$. The $\mu_j(x,t)$ live on two-sheeted Riemann surfaces; the branch points connecting them are called "band edges" E_{2j} and E_{2j+1}. The μ_j lie in the intervals (E_{2j},E_{2j+1}) and oscillate exactly j times between these limits as x is varied on $(0,L)$. When μ_j reaches a band edge the σ_j changes sign and the motion moves to the adjacent Riemann sheet. The temporal evolution of the μ_j is described by

$$\frac{d\mu_j}{dt} = 2[-u(x,t) + 2\mu_j]\mu_j' , \quad (6)$$

where $\mu_j' = d\mu_j/dx$ is given by (4) and $u(x,t) = \lambda\eta(x,t)$ is given by (3). The space (4) and time (6) ODEs evolve the $\mu_j(x,t)$ and the nonlinear Fourier series (3) constructs general spectral solutions to periodic KdV.

Each hyperelliptic function $\mu_j(x,t)$ may be written in terms of phases $\Phi_j(x,t)$, so that $\mu_j(x,t) = \mu_j[\Phi_j(x,t)]$, where $\Phi_j = K_j x - \Omega_j t + \phi_j$ [11-13]. (The Φ_j are also the phases of the associated theta-function inverse problem for KdV.) The K_j and Ω_j are the wave number and frequency, related by a dispersion relation $\Omega_j(K_j)$; the ϕ_j are constant phases [15]. To compute the Φ_j from the hyperelliptic functions $\mu_j(x,t)$ one introduces the Abelian differentials

$$d\Omega_j(E) = \sum_{k=0}^{N-1} C_{kj} \frac{E^k dE}{R^{1/2}(E)}, \quad 1 \leq j \leq N . \quad (7)$$

C_{kj} is a coefficient matrix with suitable normalization; an explicit form for the coefficients C_{kj} is readily found [15]. One then computes the phases by [11-13]

$$\Phi_j(x,t) = \sum_{m=1}^{N} \oint_{E_{2m}}^{P_m(x,t)} d\Omega_j(E) = K_j x - \Omega_j t + \phi_j , \quad (8)$$

where $P_m(x,t) = [\mu_m(x,t), \sigma_m]$. Explicit expressions for the K_j, Ω_j in terms of the coefficients C_{kj} and the main spectrum E_i are given elsewhere [15].

To obtain the phases ϕ_j set $x=0$, $t=0$ in [8] such that the upper limit becomes $P_k(0,0) = [\mu_k(0,0), \sigma_k]$ and $\phi_j = \Phi_j(0,0)$. Therefore the constant phases ϕ_j of the $\mu_j(x,t)$ depend explicitly upon the *initial values of the hyperelliptic functions $\mu_j(0,0)$, the Riemann sheet indices σ_j, and the main spectrum E_i*. These are the necessary ingredients for integrating the space ODEs (4).

The determination of the main spectrum (E_i, $1 \leq i \leq 2N+1$) and the auxiliary spectrum [$\mu_j(0,0)$, $\sigma_j = \pm 1$, $1 \leq j \leq N$] is referred to as the *direct* scattering problem. The determination of the hyperelliptic functions $\mu_j(x,t)$ by the solution of the nonlinear ODEs (4)-(6) and the construction of solutions of the KdV equation by (3) constitutes the *inverse* scattering problem. The numerical methods used herein for computing the main and auxiliary spectra, the hyperelliptic functions, and the space-time evolution of the KdV equation are given elsewhere [14-20]. It should be emphasized that the computations are made over all $N = 256$ hyperelliptic modes in the example given below.

The procedure for computing a solution to KdV is as follows [15,19]: (a) Pick the hyperelliptic function amplitudes A_j ($= E_{2j+1} - E_{2j}$, $1 \leq j \leq N$) from a known spectrum. (b) Pick the hyperelliptic function moduli, m_j. For solitons, choose values near 1, $m_j \sim 0.99999$ and for radiation choose values which are small, $m_j \sim 0.1$. (c) Compute the G_j the "gap amplitudes," $G_j = A_j(1-m_j)/m_j$. Numerically iterate on the G_j to force the wave numbers K_j to be *commensurable* as in [15]. This insures that $\eta(x,t)$ will be periodic in space and almost periodic in time. (d) Select E_1 arbitrarily [15,19]. Construct the main spectrum E_i ($i = 2, 2N+1$) by $E_{2j} = E_{2j-1} + G_j$, $E_{2j+1} = E_{2j} + A_j$ ($j = 1,N$). (e) Pick the phases ϕ_j on $(0,\pi)$ and the Riemann sheet indices, $\sigma_j = \pm 1$. Then compute the starting values of the hyperelliptic functions $\mu_j(0,0)$ by *cycle integral inversion* [see discussion and (9)-(11) below]. (f) Construct the hyperelliptic functions $\mu_j(x,0)$ by integrating the spatial ODEs (4) with initial conditions $\mu_j(0,0), \sigma_j$. (g) Construct the solution of KdV at $t=0, \eta(x,0)$, by linear superposition (3) of the hyperelliptic functions, $\mu_j(x,0)$. (h) Integrate the realization forward in time by (6) to obtain the $\mu_j(x,t)$ and use (3) to get $\eta(x,t)$.

In what follows I view the above analytical formulation of IST as a random process for the KdV equation by assuming that the parameters (ϕ_j, σ_j) are random numbers. One way to obtain *random phase realizations* of the KdV equation is to select the phases ϕ_j as uniformly distributed random numbers [in analogy with (1)]. Realizations may then be constructed using the linear superposition law (3) together with numerical integrations of (4)-(6). Since the nonlinear ODEs require the $\mu_j(0,0)$ as initial values, we need a way to invert (8) to obtain $\mu_j(0,0)$ from the randomly chosen ϕ_j. To this end I assume that there is no overall bias for one Riemann sheet over the other so that the $\sigma_j = \pm 1$ are chosen by a simple coin toss; these values are assumed known in the inversion process. Write (8) in the form (with $x=0$, $t=0$):

$$f_j(\mu_m; \phi_m, \sigma_m) = \sum_{m=1}^{N} \sum_{n=0}^{N-1} D_{mn}(\mu_m, \sigma_m) C_{nj} - \phi_j = 0 , \quad (9)$$

where

$$D_{mn}(\mu_m, \sigma_m) = \oint_{E_{2m}}^{P_m(0,0)} \frac{E^n dE}{R^{1/2}(E)} . \quad (10)$$

The goal is to solve (9),(10) for $\mu_m(0,0)$ given the ϕ_m and the σ_m; the numerical solution is accomplished by Newtonian iteration. Given starter values of the $\mu_j^{(1)}(0,0)$ (typically an adjacent band edge, E_{2j} or E_{2j+1}), iterate the algorithm:

$$\mu_j^{(i+1)} = \mu_j^{(i)} + \sum_{m=1}^{N} \left[\left(\frac{\partial f_j}{\partial \mu_m} \right)_{jm}^{-1} \right] f_m(\mu_j^{(i)}) , \qquad (11)$$

where the derivatives have an explicit analytical form [19].

The following observations give some perspective about KdV random processes: (1) The recipe [(a)-(h)] is guaranteed to generate exact spectral solutions to the KdV equation in the sense of IST; each set of random phases (ϕ_j, σ_j) generates a different realization of $\eta(x,t)$. (2) In the linear, small-amplitude limit the formulation reduces identically to (1), as required. (3) The Floquet band structure is preserved for each realization, i.e., the hyperelliptic function amplitudes are constants of the motion, just as the sine-wave amplitudes are constants in (1). (4) The realizations computed here are forced to be periodic by constructing a set of commensurable wave numbers $K_j = k_j = 2\pi j/L$ [15]. However, commensurability is not a requirement, even in (1) [21]. (5) KdV is a completely integrable Hamiltonian system with an infinite number of conservation laws. The hyperelliptic basis functions always remain inside their respective open bands and the motion lies on a $2N$ dimensional torus. As a result solutions computed by this procedure are realizations of a *stationary and ergodic random process* [19]. (6) The realizations given herein are *nonlinear fractal fields* (or *nonlinear colored random noise*) in the sense discussed in [5,22] with dimension $D = 2/(\gamma - 1) \sim 2$. (7) Extension of the present approach to the theta-function inverse problem is straightforward (and equivalent to that given here) [19], although several numerical advantages are obtained from the μ representation [20].

The procedure actually used here for determining the $\{E_i\}$ is slightly different than the general approach just outlined [steps (a)-(d)]. Equation (1) is implemented to compute a linear realization of the function $u(x,t) = \lambda\eta(x,0)$. The power law is taken to be $\gamma = 2.0$ with nonlinearity parameter $\lambda = 0.012$; this case has the same ratio of dispersion to nonlinearity, $\delta = 0.022$, as used in ZK. The IST spectrum is then determined from the space series, $\eta(x,0)$, as though it were a Cauchy initial condition for KdV. Reasons for using this modified procedure here are (a) to obtain an IST spectrum which differs somewhat from a perfect power law, with amplitude variations hopefully similar to what one might obtain in an experimental situation, and (b) to determine what the KdV equation thinks the hyperelliptic spectrum is for a perfect linear Fourier power law. Given the main spectrum $\{E_i\}$ determined in this way [16,20] the computed IST phases have been replaced with random numbers as described above and the realization $\eta(x,0)$ seen in Fig. 1(a) is then constructed [steps (e)-(g)]. The amplitudes of the inverse scattering (hyperelliptic function) modes are shown in Fig. 1(c), together with the linear input power spectrum k^{-2} (note that the spectral amplitudes, not the power spectral amplitudes, are graphed).

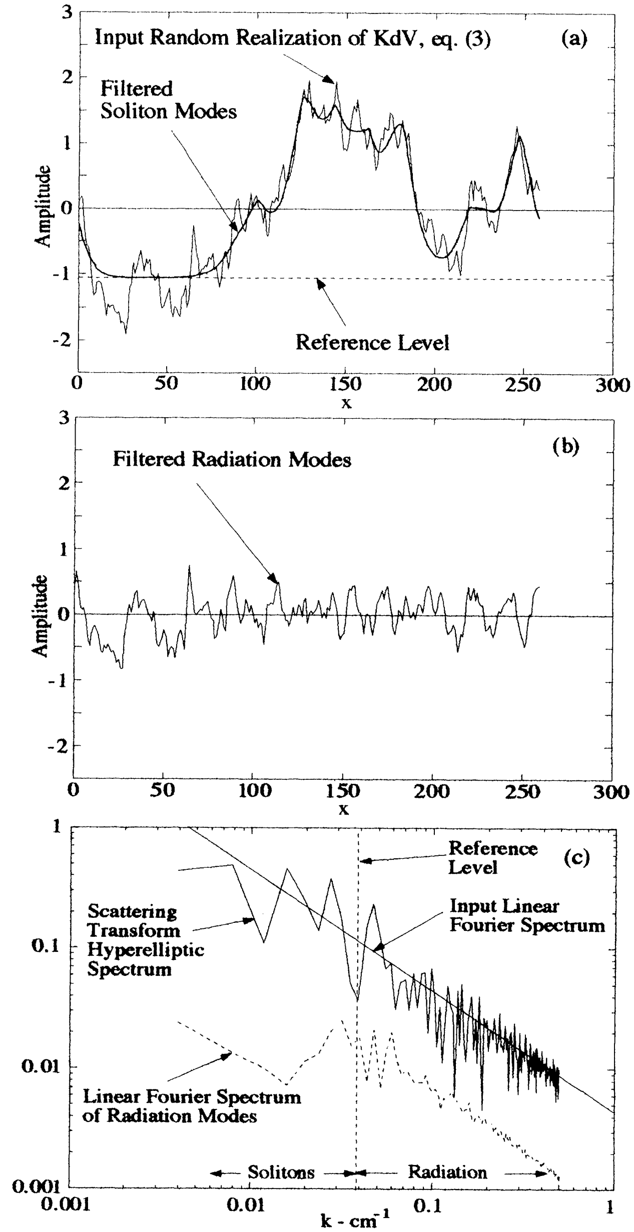

FIG. 1. (a) Realization of nonlinear random process for the KdV equation, $\gamma = 2.0$, $\lambda = 0.012$. Also shown are the soliton components as filtered from the realization. (b) The filtered radiation components. (c) The hyperelliptic-function amplitude spectrum; the input Fourier spectrum and the linear Fourier transform of the radiation modes (arbitrarily shifted downward for visual clarity).

The inverse scattering mode amplitudes are scattered above the input power-law Fourier modes, presumably due to nonlinear interactions [Fig. 1(c)]. The first nine of the KdV modes (at low wave number) are found to represent solitons propagating on a "reference level" which is

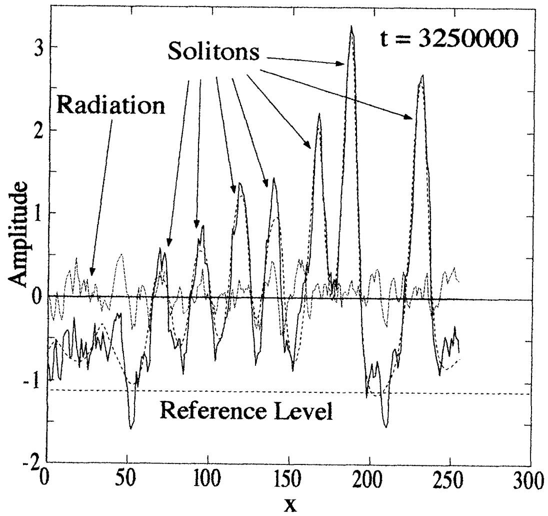

FIG. 2. Example in Fig. 1 evolved forward in time to $t=3250000$ (solid line). Individual soliton peaks are now visible, whereas they are not in the initial realization $\eta(x,0)$ in Fig. 1(a). Solitons (bold dotted line) are determined from the IST by filtering out the radiation. Radiation (light dotted line) is determined by filtering out the solitons.

below the mean level [Fig. 1(a)] [14], while the remaining (high wave number) modes represent the radiation contribution. IST is then used to isolate the soliton components of the realization [i.e., only the soliton part of the linear superposition law (3) is summed, a process called *nonlinear filtering* [6,20]]; the resultant soliton components of the wave train can be seen in Fig. 1(a). The radiation components are shown in Fig. 1(b); these are found by nonlinearly filtering out the nine solitons by summing (3) only over the radiation modes in the wave train. The linear Fourier transform of *the radiation modes* shown in Fig. 1(b) is given in Fig. 1(c); the results suggest that the power-law behavior in the spectrum is preserved at high wave numbers, but substantial energy is still found at lower wave numbers.

Since the solitons are not easily visible in the initial realization [Fig. 1(a)], an interesting question is: Are the solitons *ever* visible during their space-time evolution? To investigate this I integrated the motion forward in time by (3)-(6) to $t=3250000$ s and found the results given in Fig. 2; shown are the integrated realization, the filtered soliton components, and the filtered radiation modes. One sees that the largest peaks are found to be solitons, as verified by the inverse scattering transform.

I acknowledge valuable council from L. Bergamasco. This work was supported in part by the Office of Naval Research (ONR Grant No. N00014-92-J-1330) and by the Progetto Salvaguardia di Venezia del Consiglio Nazionale delle Ricerche, Italy.

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