Routes to Suppressing Chaos by Weak Periodic Perturbations

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A new route is described for eliminating chaos in nonlinear oscillators by changing only the shape of a weak nonlinear periodic perturbation and illustrated with the example of the Duffing-Holmes oscillator forced with the Jacobian elliptic function sn. Two techniques are used in the illustration: applying the Melnikov-Holmes analysis, and studying the behavior of the Lyapunov exponent from a simple recursion relation which models an unstable limit cycle. The connection with related previously described routes is also discussed in a general setting.

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During the last decades, new phenomena have appeared from the inclusion of terms modeling periodic perturbations in the equations of nonlinear dissipative systems. The most ubiquitous is chaos [1]. It is also clear at first sight from the literature that it is the harmonic functions which have been overwhelmingly used to model the periodic perturbations. However, these functions are solutions of *linear* oscillators, and rarely of nonlinear equations. Nature is nonlinear, so we should also take the forcing mechanism to be a nonlinear system since this is the generic situation. In other words, it seems more appropriate to employ periodic functions that are solutions of *nonlinear* oscillators to construct more realistic perturbations. The simplest functions having this requirement are the Jacobian elliptic functions (JEF) [2]. If one considers polynomials to be the simplest nonlinear extension of linear oscillators, their solutions are known to be given in terms of JEF's. This is the case for the most studied nonlinear integrable oscillators, such as the Duffing or the Helmholtz. Also for nonlinearities in the form of harmonic functions, the pendulum, for instance [3], the solutions are in terms of JEF's. In comparison with the harmonic solutions, the JEF's add a new variable to the parameter space of the system: the elliptic parameter m that is responsible for the *shape* of the perturbation, i.e., for the temporal rate at which energy is transferred from the excitation mechanism to the system, having fixed the period. This fact leads us to expect new aspects of behavior of the system—unexplored in the harmonic case—when m is varied, the remaining parameters being left constant.

In this Letter, we show how by altering solely the shape of a weak external nonlinear modulation, one may pass a dynamical system from a regular to a chaotic state, and vice versa. That suggests a possible explanation of some proposed mechanisms for controlling chaos. From previous work on the possibility of eliminating or reducing chaos in a dynamical system [4,5], it seems that the *resonant* property of the *harmonic* perturbation causing the regularization of the system is a necessary condition for a complete regularization. In particular, this is the case for the model

$$\ddot{x}+f(x,\dot{x})=A\sin(\omega t)+\alpha\sin(\beta\omega t)\,, \tag{1}$$

where $\alpha\sin(\beta\omega t)$ is responsible for the disappearance of chaos when α and β are suitably chosen—starting from chaos at $\alpha=0$. [See Ref. [4] where $f(x,\dot{x})=\sin x+G\dot{x}-I$; G,I constants.] The resonance condition implies that the two terms on the right-hand side (rhs) of Eq. (1) would belong to the *same* Fourier expansion of a periodic function if A and α fit some given conditions. Such a periodic function should be closely related to some resonant steady periodic solutions of the associated Hamiltonian system—if one is looking for a regular response of the whole equation. Observe that underlying the above scheme is the generalized nonlinear version (elliptic forcing) of harmonically forced self-exciting systems [6]. Also, both steady and transient solutions of the most physically relevant, nonchaotic, nonlinear oscillators are generally given in terms of JEF's [7,8]. We are thus proposing that the inhibition or reduction of an initial chaotic state depends in a fully nonlinear situation on the three parameters of the periodic perturbation causing the phenomenon: period, amplitude, and shape (besides, of course, the initial conditions and the remaining parameters of the system).

To look at the foregoing ideas in a concrete model, we selected the Duffing-Holmes oscillator [9], partly because its chaotic transition can be predicted, albeit approximately, by Melnikov-Holmes analysis (MHA), and partly because this model has been widely studied [1]. The equation is

$$\ddot{x}-x+\alpha x^{3}=-\delta\dot{x}+\gamma\,\mathrm{sn}(\omega t;m)\,, \tag{2}$$

where $\delta,\gamma\ll 1$, and $\mathrm{sn}(\omega t;m)$ is the JEF of parameter m. When $m=0$, then $\mathrm{sn}(\omega t;m=0)=\sin(\omega t)$; i.e., we recover the extensively studied case of harmonic forcing [6,9]. This is fundamental in comparing the structural stability

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of the system when only the forcing shape is varied. To this end, we fix the forcing period T=const, making the frequency $\omega(m)\equiv 4K(m)/T$, where K is the elliptic integral of the first kind. In this situation, the parameter space of Eq. (2) is four dimensional due to the addition of the parameter m to the three-dimensional parameter space (α,δ,γ) of the corresponding harmonic counterpart and we have in the other limit

$$\mathrm{sn}(\omega t;m=1)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin[(2n+1)2\pi t/T]\,.$$

Note that this is the Fourier expansion of the *square wave* function of period T whose shape is not *exactly* a square wave, due to the well-known *Gibbs' phenomenon* [10]. In spite of this we can approach Eq. (2) in the limit $m=1$ by writing

$$\begin{aligned}&\ddot{x}-x+\alpha x^3+\delta\dot{x}-\gamma=0,\quad 0<t<T/2\ (\mathrm{mod}T)\,,\\&\ddot{x}-x+\alpha x^3+\delta\dot{x}+\gamma=0,\quad T/2<t<T\ (\mathrm{mod}T)\,.\end{aligned}\tag{2'}$$

Now, we choose the parameters α, δ, and γ to be constant, and vary m from 0 to 1 to study the pure effect of variations in the shape of the perturbation. We are here mainly interested in the stability of chaos under these changes, so we apply MHA to Eq. (2). This method is now a standard procedure [9,11]. To quote Brunsden and Holmes [12]: "This analytical method detects transverse homoclinic points in differential equations which are small perturbations of integrable systems. This, with the Smale-Birkhoff homoclinic theorem [9,13], implies the existence of chaotic motions among the solutions of the equations in question. . . ." This means, among other things, that only *necessary* conditions for chaos are obtained from MHA, and therefore one may always get *sufficient* conditions for the suppression of chaos by using this tool. This is the principal foundation of the utility of MHA in predicting the inhibition of chaotic behavior [14].

For Eq. (2) the Melnikov distance is

$$\Delta(t_0)=\int_{-\infty}^{+\infty}\{-\delta u_s^2(t)+\gamma u_s(t)\,\mathrm{sn}[\omega(t+t_0);m]\}dt\,,\tag{3}$$

where

$$x_s(t)=(2/\alpha)^{1/2}\mathrm{sech}t,\quad u_s(t)=-(2/\alpha)^{1/2}\mathrm{sech}t\tanh t\,,\tag{4}$$

are the parametric equations of the homoclinic loop corresponding to the hyperbolic fixed point of the Duffing-Holmes equation with $\delta=\gamma=0$. Using the Fourier expansion of sn [2], and after some simple algebraic manipulation, Eq. (3) can be recast into the form

$$\begin{aligned}\Delta(t_0)=&-(2\delta/\alpha)\int_{-\infty}^{+\infty}\mathrm{sech}^2\tau\tanh^2\tau\,d\tau-\gamma(2/\alpha m)^{1/2}(\pi/K)\sum_{n=0}^{\infty}\mathrm{csch}[(n+1/2)\pi K'/K]\cos[(n+1/2)\pi\omega t_0/K]\\&\times\int_{-\infty}^{+\infty}\mathrm{sech}^2\tau\sinh\tau\sin[(n+1/2)\pi\omega\tau/K]d\tau\,,\end{aligned}\tag{5}$$

with K' the complementary complete integral of the first kind. The resulting integrals can be evaluated from standard integral tables [15]. Finally, we obtain

$$\Delta(t_0)=-(4\delta/3\alpha)-\gamma\frac{\omega\pi^3}{K^2(2\alpha m)^{1/2}}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{csch}\left[(n+1/2)\frac{\pi K'}{K}\right]\mathrm{sech}\left[\frac{(2n+1)\pi^2\omega}{4K}\right]\cos\left[\frac{(2n+1)\pi\omega t_0}{2K}\right].\tag{6}$$

From Eq. (6), with $\omega(m)\equiv 4K(m)/T$, it is straightforward to demonstrate that a homoclinic bifurcation is guaranteed for trajectories whose initial conditions are sufficiently near the unperturbed separatrix (4) if

$$\delta/\gamma<U(\alpha,m,T)\,,\tag{7}$$

where the threshold function is

$$U(\alpha,m,T)=\frac{3\sqrt{2}\pi^3}{2TK}\left[\frac{\alpha}{m}\right]^{1/2}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{csch}\left[(n+1/2)\frac{\pi K'}{K}\right]\mathrm{sech}\left[\frac{(2n+1)\pi^2}{T}\right].\tag{8}$$

With T and α constant, we study the chaotic threshold as a function of only the forcing shape parameter m. A typical plot of $U(m)$ is shown in Fig. 1. The qualitative form of this function remains the same as α and T are varied. It is illuminating to consider the limit case $m=1$. From Eq. (8) we obtain

$$U(\alpha,m=1,T)=(6\pi\sqrt{2\alpha}/T)\sum_{n=0}^{\infty}\mathrm{sech}[(2n+1)\pi^2/T]\,,\tag{8'}$$

with $\lim_{t\to\infty}U(\alpha,m=1,T)=0$; i.e., in this limit chaotic behavior is not possible. This result is consistent with well-known results concerning the related harmonic case [9]. Thus, observe that when $T\to\infty$ the system (2′) is well represented by the equation $\ddot{x}-x+\alpha x^3+\delta\dot{x}=\gamma$, which is likewise the limit of the familiar system $\ddot{x}-x+\alpha x^3+\delta\dot{x}=\gamma\cos\omega t$ when $\omega\to 0$. For this equation the threshold function is [9] $U'(\alpha,\omega)=(3\sqrt{2\alpha}/4)\pi\omega\,\mathrm{sech}(\pi\omega/2)$ with $\lim_{\omega\to 0}U'(\alpha,\omega)=0$, and we find again the impossibility for the onset of chaos. Also, for $m=0$ we have, from Eq. (8),

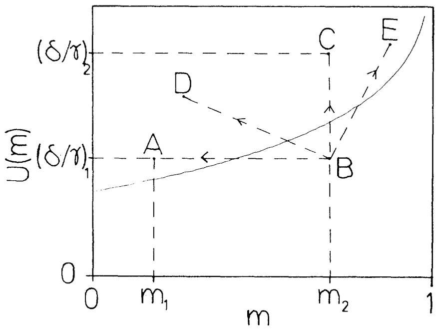

FIG. 1. Threshold function $U(m)$ versus m [Eq. (8)] in a generic situation, the remaining parameters being held constant.

$$U(\alpha,m=0,T)=(3\pi^2\sqrt{2\alpha}/2T)\,\mathrm{sech}(\pi^2/T)\,. \tag{8''}$$

In summary, the threshold functions $U(\alpha,m,T)$ for fixed α, thought of as functions of T only, have the same qualitative shape for every $m\in[0,1]$: They are positive functions with only one maximum, and satisfy $\lim_{T\to 0,\infty}U(\alpha,m,T)=0$. Straightforward calculation shows that the degree of agreement between the theoretical chaotic threshold, Eq. (8), and numerical experiments is the same for $m\to 1$ as for other m values. In other words, the case $m\to 1$ (square wave) is not qualitatively different from the harmonic case ($m=0$).

Let us suppose that the system (2) is in a regular state denoted by A (see Fig. 1). Then, increasing m from m_1 to m_2, and keeping constant the remaining parameters α, T, δ, and γ, the system may reach a state (point B) *capable* of being chaotic. Contrariwise, if B represents a steady chaotic state, the route $B\to A$ regularizes such a state. Now, fixing α, T, and m we can increase the ratio δ/γ (raising δ, decreasing γ, or both) reaching a regular state at point C: This is a known procedure for taming chaos [9,11]. Note that the pathways of types $B\to A$ and $B\to C$ are only *ad hoc* routes to regular states, the most common being a simultaneous variation of δ/γ and m, as in the paths $B\to D$ and $B\to E$. In our opinion, this is exactly the scenario behind the inhibition of chaos by weak harmonic perturbations. Indeed, in Eq. (1) the resonant condition implies that $\beta=n/m$, $n,m\in Z$, i.e., both sines have a *common* period $T=2\pi m/\omega$ and the addition of $\alpha\sin(\beta\omega t)$ to $A\sin(\omega t)$ has the effect of varying both the amplitude and the shape of the original forcing term. Therefore, it is possible that taking the appropriate α value in addition to the resonance condition the system may reach a regular state (as observed in numerical experiments, cf. Ref. [4]).

The relevance of the coexistence of infinitely many periodic unstable solutions is today quite clear, and is considered synonymous with steady chaos (strange attractor), so that we may test the above scheme by the model of an unstable limit cycle affected by a weak periodic perturbation given by the JEF sn:

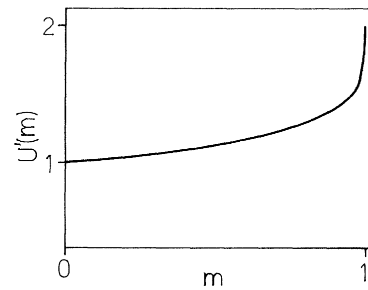

FIG. 2. Function $U'(m)$ versus m [Eq. (11)].

$$x_{n+1}=(\mu+\epsilon f_n)x_n\,, \tag{9}$$

with $\mu>1$, $f_n=\sqrt{2}\,\mathrm{sn}(2K/\pi n;m)$, i.e., for simplicity, we have chosen the period $T=2\pi$. A similar recursion relation with f_n a harmonic function is considered in Ref. [4]. Note that $\langle f_n\rangle=0$, and $\langle f_n^2\rangle=(2/m)(1-E/K)$, with E the complete elliptic integral of the second kind, angular brackets denoting the average over n. If $m\to 0$, $\langle f_n^2\rangle\to 1$ as in the harmonic counterpart. When $\epsilon=0$, the fixed point x is unstable. To study the effect of the weak modulation, we calculate the Lyapunov exponent (LE) for $\epsilon\neq 0$:

$$\lambda=\mathrm{Re}\langle\ln(\mu+\epsilon f_n)\rangle\,. \tag{10}$$

If the limit cycle is weakly unstable $\mu=1+|\delta|$, $|\delta|\ll 1$. In this situation, for small ϵ, Eq. (10) becomes $\lambda=|\delta|-\epsilon^2U'(m)+O(\epsilon^3)$ with

$$U'(m)=\frac{2}{m}\left[1-\frac{E}{K}\right]. \tag{11}$$

A plot of $U'(m)$ is presented in Fig. 2 (observe the similarity with Fig. 1). When $|\delta|<\epsilon^2U'(m)$, the LE λ is negative, i.e., x is stable. On the contrary, if $|\delta|>\epsilon^2U'(m)$, λ is positive and x is unstable. In order to clarify the effect of shape on the reduction of instabilities (positive LE), let us consider that we are in an initial state characterized by $\epsilon=\epsilon_1$, $m=m_1\sim 0$ such as $|\delta|>\epsilon_1^2U'(m_1)$. Then, by increasing m, the LE λ decreases and, in some case, may become negative, thus stabilizing x.

A complete understanding of all this would, of course, involve obtaining resonant periodic solutions of the general problem

$$\ddot{x}+f(x,\dot{x})=A\,\mathrm{pq}(\omega t;m)\,, \tag{12}$$

where $f(x,\dot{x})$ is a nonlinear function, and pq is a JEF closely connected to the solutions of the associated Hamiltonian system. In a forthcoming paper, we will present a more detailed study, including numerical experiments.

The analysis based on Eq. (2) can be developed in three important ways [16]. First, the results extend to general modulated dynamical systems $x = F(x)$ near the onset of chaos. Second, the modulation $\gamma \mathrm{sn}(\omega t;m)$ need not enter additively, as in Eq. (2), but can enter instead as a *parametric* modulation. Third, the parameter m extends the codimension-1 bifurcation (saddle-node, transcritical, pitchfork, period-doubling, and Hopf, typically encountered as a single control parameter is varied) to new codimension-2 bifurcations.

In summary, we have presented a new way to reduce or suppress steady chaotic states, by only altering the geometrical shape of weak periodic perturbations. We connected it with related known mechanism in a general context. Finally, using a simple model recursion relation, we found the same new route to order.

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