Growth Rate of the Richtmyer-Meshkov Instability at Shocked Interfaces

Karnig O. Mikaelian
University of California, Lawrence Livermore National Laboratory, Livermore, California 94550
(Received 16 July 1993)

We have found several cases in our numerical simulations where Richtmyer's prescription fails to give the correct growth rate for the Richtmyer-Meshkov instability in the linear regime. Another expression, due to Fraley [Phys. Fluids **29**, 376 (1986)], agrees with our simulations. We discuss recent experiments and report two new types of perturbation freeze-out, single-shock and double-interface, which are quite distinct from the previously reported double-shock freeze-out.

PACS numbers: 47.40.Nm, 47.20.Ma, 52.35.Py

The Richtmyer-Meshkov (RM) instability [1,2] refers to the growth of perturbations at a shocked interface. Recently RM instabilities have been the subject of intense theoretical, computational, and experimental research because of their importance in inertial-confinement-fusion implosions as well as supernova explosions [3]. They are a general feature of shocked interfaces and occur for shocks moving from a low to a high density fluid or vice versa.

Starting with Meshkov [2], experimental results have been compared with Richtmyer's expression [Eq. (1) below] for the growth rate of the instability in the linear regime. There is qualitative agreement, but quantitatively the experimental growth rates are smaller, sometimes by a factor ~ 2. Numerical simulations, on the other hand, have so far agreed with Richtmyer, the first simulations being reported by Meyer and Blewett [4]. More sophisticated code calculations can be found in Ref. [3], with the result that in the linear regime the normalized growth rate (R_{NG}) of the instability is [1]

$$R_{NG} \equiv \frac{\dot{\eta}}{\eta_0 \Delta v k} = A_{\text{eff}} = \left(1 - \frac{\Delta v}{W_i}\right) A_{\text{after}} . \tag{1}$$

In the above equation $\eta(\tau)$ is the amplitude of the perturbation (τ is time), η_0 is its initial value, Δv is the jump velocity imparted by the shock, $k = 2\pi/\lambda$ with λ the wavelength of the perturbation, W_i is the speed of the incident shock wave, and A_{after} is the Atwood number of the fluids after the shock. The factor $1 - \Delta v/W_i$ in Eq. (1) is the compression factor relating η_{after} to η_0, where η_{after} is the amplitude immediately after the shock, i.e., $\eta_{\text{after}} = (1 - \Delta v/W_i)\eta_0$ [see Eq. (50) in Ref. [1]]. In the incompressible limit the compression factor $\to 1$ and $A_{\text{after}} \to A_{\text{before}}$, the Atwood number before the shock,

$$A_{\text{before}} = \frac{\rho_B - \rho_A}{\rho_B + \rho_A} = \frac{R-1}{R+1}, \quad R \equiv \rho_B/\rho_A , \tag{2}$$

where ρ_A and ρ_B are the densities of the fluids before the shock. Our convention is that the shock is initiated in fluid A and proceeds towards fluid B. Richtmyer first derived the incompressible result, $A_{\text{eff}} = A_{\text{before}}$, by treating the shock as an instantaneous acceleration. After a number of calculations with the linearized (meaning $\eta k \ll 1$) but otherwise fully compressible perturbation equations, he gave Eq. (1) as a prescription for the growth rate attained by the perturbations after a brief transient period. The fact that $\dot{\eta}$ is a constant in time leads directly to perturbations growing linearly in time,

$$\eta(\tau) = \eta_0 [1 + \Delta v k A \tau] , \tag{3}$$

where $A = A_{\text{before}}$ in the incompressible limit and, more generally, A_{eff} for compressible fluids.

In the majority of our simulations, including weak as well as strong shocks, we found Eq. (1) to be a good prescription for $\dot{\eta}$, following earlier findings [3,4]. However, we also found a number of cases, again including weak as well as strong shocks, where Eq. (1) failed. Before we elaborate on those cases we emphasize two points: First, *the failure is not associated with the linear approximation*—we took extremely small amplitudes. Second, *none of those cases involve experiments that have already been performed.* In fact, when we analyzed the experiments of Meshkov [2] and the more recent experiments of Benjamin [5] (see below), we found extremely good agreement with Eq. (1). New experiments will have to be done to see the failure of Richtmyer's prescription.

The most conspicuous case involved $A_{\text{after}} = 0$ which, according to Eq. (1), should have led to $\dot{\eta} = 0$, i.e., freeze-out of the perturbation by a single shock (to differentiate from double-shock freeze-out expected to occur in compressible as well as incompressible fluids, as reported previously [6].) We found cases where $A_{\text{after}} = 0$ yet $\dot{\eta} \neq 0$. We started with compressible fluids A and B, $\rho_B > \rho_A$, so that $A_{\text{before}} > 0$. The incoming shock was tuned to give $A_{\text{after}} = 0$. However, we found $\dot{\eta} > 0$, in direct violation of Eq. (1).

The strong shocks require solving a transcendental equation to arrive at postshock quantities like A_{after}, etc. (details can be found in Ref. [7]). Here we concentrate on weak shocks because explicit expressions can be obtained for $\dot{\eta}$. Defining the overpressure ratio $\varepsilon = (p_3 - p_0)/p_3$ we keep only terms linear in ε. Here p_3 (p_0) refers to the pressure behind (ahead) of the incident shock whose Mach number is usually denoted by M_s, related to ε via $M_s^2 = 1 + \varepsilon(\gamma_A + 1)[2\gamma_A(1-\varepsilon)]^{-1}$. We assume ideal equations of state and $\gamma_{A,B}$ stand for the constant adiabatic indices of fluids A and B.

0031-9007/93/71(18)/2903(4)$06.00

To first order in ε we find

$$1-\frac{\Delta v}{W_i}=1-\frac{2\varepsilon}{\gamma_A(1+y)}\,, \tag{4a}$$

and

$$A_{\text{after}}=A_{\text{before}}+\frac{4\varepsilon yR}{\gamma_A\gamma_B(1+y)(1+R)^2}(\gamma_A-\gamma_B)\,, \tag{4b}$$

where

$$y\equiv\sqrt{\rho_B\gamma_B/\rho_A\gamma_A}=\sqrt{R\gamma_B/\gamma_A}\,. \tag{4c}$$

Equation (4a) shows that the compression factor <1 as expected; Eq. (4b) shows that, to first order in ε, $A_{\text{after}}=A_{\text{before}}$ if $\gamma_A=\gamma_B$, and that $A_{\text{after}}>$ or $<A_{\text{before}}$ depending on whether $\gamma_A>$ or $<\gamma_B$. The latter are the interesting cases.

Combining Eqs. (4) and (1) we obtain

$$A_{\text{eff}}=A_{\text{before}}+\varepsilon F/\gamma_A\,, \tag{5}$$

where

$$F(R,y)=2\left[1-R+2\frac{R(R-y^2)}{y(R+1)}\right]\times(R+1)^{-1}(y+1)^{-1}\,. \tag{6}$$

Perhaps most interesting is the case $A_{\text{before}}=0$ where the fluids differ only by virtue of their different γ's, not densities. Then $R=1$ and

$$A_{\text{eff}}=0-\frac{\varepsilon}{\gamma_A}(y-1)/y,\quad y=\sqrt{\gamma_B/\gamma_A}\,. \tag{7}$$

However, when we ran a test problem with $\varepsilon=0.4$, $\gamma_A=1.1$, $\gamma_B=4.4$ ($y=2$), we found $\dot\eta=0$, in contrast to Eq. (7), according to which $R_{\text{NG}}=A_{\text{eff}}=-0.18$; i.e., perturbations should have grown after a phase change. We emphasize that the compression of the perturbation as well as the Atwood number after the shock were correctly given by Eqs. (4a) and (4b). Only their product failed to give the correct growth rate, violating Richtmyer's prescription.

These and other failures led us to consider an analysis by Fraley [8] who, to first order in ε, gave an expression similar to Eq. (5). His function F, however, differs from Eq. (6) and is

$$F(R,y)=\left[(y-1)^2+4\frac{R^2+y^2}{y(R+1)}-2R-2y\right]\times(R+1)^{-1}(y+1)^{-1}\,. \tag{8}$$

[We have corrected a misprint in Fraley's Eq. (49)]. For the case under consideration we set $R=1$ so Eq. (8) gives

$$A_{\text{eff}}=0-\frac{\varepsilon}{\gamma_A}(y-1)(1-0.5y)/y,\quad y=\sqrt{\gamma_B/\gamma_A}\,, \tag{9}$$

to be compared with Eq. (7). The extra factor $1-0.5y$ successfully explains the lack of growth for $y=2$ reported above.

In general, Eq. (8) explains well the behavior of perturbations in weak shocks. While $\dot\eta$ for strong shocks cannot be given in closed form, Fraley also gave a fourth-order expression, too complicated to be reproduced here, which we found to agree well with our simulations. We also explain (see Ref. [7] for details) why earlier experiments, including some strong shock calculations by Richtmyer, did not show deviations from Eq. (1). We must emphasize that Richtmyer gave his last equation, Eq. (72), as a simple "recipe" or prescription that captures the effects of compressibility. He had found these effects by solving several examples using a code based on his linearized but fully compressible hydrodynamic equations. The *same* equations form the starting point for Fraley who, instead of solving them numerically, used Laplace transforms to derive analytic expressions. Richtmyer's prescription is most attractive to our physical intuition (there is none of it in Fraley's work) and for that reason Eq. (1) has been taken almost for granted. Just as exceptions often prove the rule, it is remarkable that Eq. (1) does so well in the majority of cases, notwithstanding the few exceptions we have found.

Fraley's analysis has a wider range of validity than Richtmyer's prescription. Where they overlap our numerical simulations also agree. Typically, but not exclusively, those problems involve large Atwood numbers and large γ_A and γ_B. Where they disagree, our numerical simulations side with Fraley. Typically, but again not exclusively, those problems involve freeze-out: We do not find the freeze-out expected by Richtmyer (of course he never considered such cases), and conversely we do find it where it is not expected. By siding with Fraley our hydrocode calculations point to the correctness of his analysis and, through it, the correctness of Richtmyer's linearized equations which formed the basis of that analysis. Only Richtmyer's last equation, in the form of a prescription given to circumvent the difficulties of an analytic solution, fails our tests involving freeze-out.

A variant of that prescription was proposed by Meyer and Blewett [4] for the case where a rarefaction, instead of a shock, is reflected from the interface. Their modification, based on numerical simulations of experiments by Meshkov, was to use the average of the initial and the final amplitudes. We found this to be a good prescription for many such problems. But again we found cases where this prescription failed, and again they involved freeze-out. Clearly, when $A_{\text{after}}=0$, which can be arranged when either a shock or a rarefaction is reflected, one expects $\dot\eta=0$ whether one uses the original ($\dot\eta\sim\eta_{\text{after}}A_{\text{after}}$) or the modified $[\dot\eta\sim\frac{1}{2}(\eta_{\text{before}}+\eta_{\text{after}})\times A_{\text{after}}]$ prescription. However, we found [7] examples involving rarefactions where $A_{\text{after}}=0$, yet $\dot\eta\neq0$.

Experiments.— We focus on Benjamin's air/SF_6 experiment [5] (we found similar results in Meshkov's experiments). Taking $\rho_A=1.22$ mg/cm^3, $\rho_B=6.20$ mg/cm^3, $\gamma_A=1.40$, $\gamma_B=1.09$ ($A=$air, $B=SF_6$), and $\varepsilon=0.39$ (hence $M_s=1.24$), we find $A_{\text{before}}=0.67$, $A_{\text{after}}=0.70$,

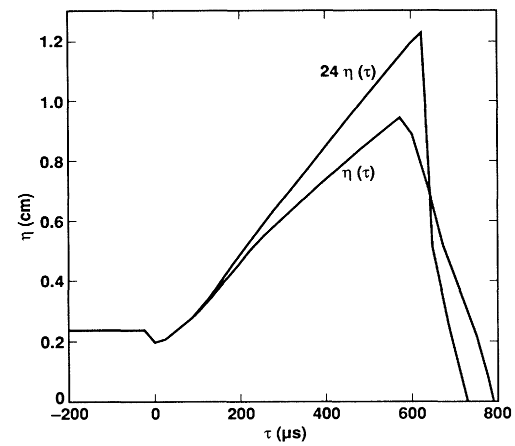

FIG. 1. $\eta(\tau)$ vs τ from 2D simulations of air/SF_6 experiments by Benjamin [5]. The lower curve started with the experimental value $\eta(0)=0.24$ cm; the upper curve is 24 times $\eta(\tau)$ from an identical run with $\eta(0)=0.01$ cm. From the slope of the curves between 300 and 600 μs we deduce that the R_{NG} is 0.37 (0.54) for the lower (upper) curve. The interface is reshocked at $\tau\sim600$ μs—see Fig. 2.

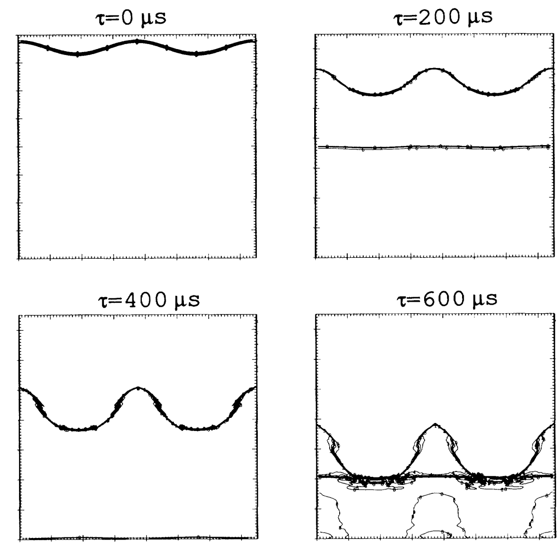

FIG. 2. Snapshots from 2D simulations of air/SF_6 experiments (see text). The test section is 7.5 cm wide and 7.5 cm high. At $\tau=200$ μs we see the interface and the transmitted shock which reflects off the lower wall at $\tau\sim400$ μs and reshocks the interface at $\tau\sim600$ μs. Snapshots after reshock are shown in Fig. 3.

compression factor$=0.81$, and therefore A_{eff}, which gives the R_{NG}, is 0.57 ($=0.81\times0.70$) according to Richtmyer's prescription and 0.58 according to Fraley's analysis, surprisingly close. Note that $A_{\text{after}}>A_{\text{before}}$ but the compression factor in Richtmyer's prescription more than compensates for the slight increase in Atwood number (the same happens in some of Meshkov's experiments). Of course both answers are below the incompressible result $R_{NG}^{\text{incomp}}=A_{\text{before}}=0.67$. We carried out a numerical simulation of this experiment first with a small amplitude, $\eta_0=0.01$ cm, obtaining $R_{NG}\approx0.54$, and second with a larger amplitude, $\eta_0=0.24$ cm (the actual experimental value) obtaining $R_{NG}\approx0.37$. The time evolution of the perturbations is shown in Fig. 1, from which the normalized growth rates can be obtained using the above mentioned values of η_0, $k=2\pi/\lambda=2\pi/3.75$ cm^{-1}, and $\Delta v=8.1$ cm/ms.

The reduction from 0.54 to 0.37 is the result of nonlinearity with the larger initial amplitude. In Benjamin's experiments R_{NG}^{expt} was ~0.24, suggesting that the membrane had some residual effect, which is expected. In summary, compression reduces R_{NG} from 0.67 to 0.58; further reduction to 0.37, from our code simulations, is ascribed to the nonlinearity of the experimental amplitude. Finally, the membrane, whose effect we cannot simulate, is probably the third and last cause for further reduction to $R_{NG}^{\text{expt}}\sim0.24$.

Snapshots from our simulations are shown in Fig. 2 for $\eta_0=0.24$ cm. In the last frame the reflected shock has returned to reshock the air/SF_6 interface. Snapshots after reshock are shown in Fig. 3, revealing highly nonlinear phenomena. The small-amplitude run, not shown here, evolved into a weakly nonlinear regime after reshock.

Finite-thickness layers.— RM experiments *without* a membrane were recently reported [9]. However, the perturbation amplitudes were even larger than the previous experiments ($\eta_0/\lambda\sim0.06$ and ~1 in Refs. [5] and [9], respectively). Here we are interested in another feature, the finite thickness of the SF_6 layer, which precludes any

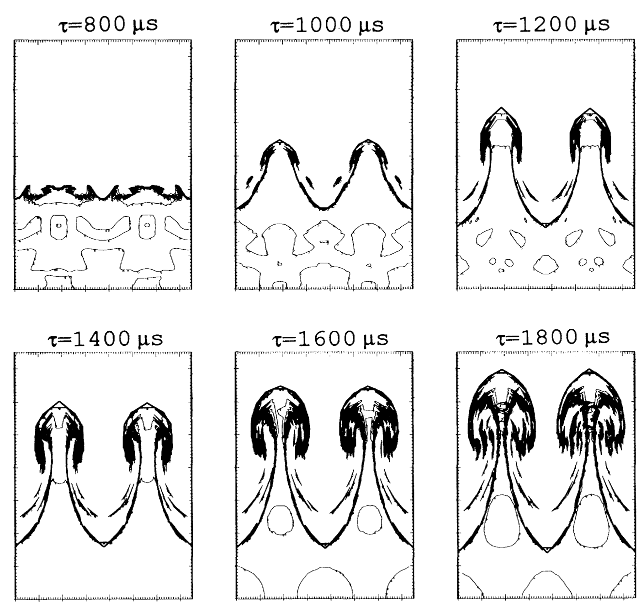

FIG. 3. Snapshots after reshock.

comparison with Richtmyer or Fraley even in the linear regime because their formulas apply only to the "classical" configuration of two semi-infinite fluids. For a finite-thickness layer one finds interface coupling whereby the evolution of the perturbation at one interface depends on the other [6].

We first introduce a "coupling angle" θ which vanishes in the classical limit (infinite thickness) and approaches $\pi/2$ in the opposite limit. Letting t stand for the thickness of the intermediate layer B in the $A/B/A$ configuration (A = air, $B = SF_6$ in the experiments), we define

$$\sin\theta = 2c/(1+c^2)\,, \tag{10}$$

where

$$c = 1 + ST + \frac{S}{R}\{1 + [1 + R^2 + 2R\coth(kt)]^{1/2}\}\,. \tag{11}$$

Here $S \equiv \sinh(kt)$ and $T \equiv \tanh(kt/2)$, and $R = \rho_B/\rho_A$ as before. For any value of R (one may even consider $R<1$), we find that $\sin\theta \to 0$ in the classical limit as $kt \to \infty$, and $\sin\theta \to 1$ as $kt \to 0$, i.e., the strong coupling limit. In the experiments [9] $kt \approx 3$ and $R \approx 5$, so we find $\sin\theta \approx 0.1$, indicating weak coupling.

In terms of θ the evolution of the perturbation at each interface is given by an exceptionally simple form (this was our reason for introducing θ):

$$\eta_1(\tau) = \eta_1(0) + \frac{\Delta v \Gamma^2}{\cos\theta}\{\eta_1(0) - \sin\theta\,\eta_2(0)\}\tau\,, \tag{12a}$$

$$\eta_2(\tau) = \eta_2(0) - \frac{\Delta v \Gamma^2}{\cos\theta}\{\eta_2(0) - \sin\theta\,\eta_1(0)\}\tau\,, \tag{12b}$$

where

$$\Gamma^2 \equiv k(R-1)/[1 + R^2 + 2R\coth(kt)]^{1/2}\,. \tag{13}$$

Our convention is that η_1 (η_2) is the first (second) interface perturbation to be shocked; i.e., interface 1 is the upstream side of the layer and interface 2 is its downstream side. In the limit $kt \to \infty$, $\Gamma^2 \to kA$ and Eqs. (12a) and (12b) reduce to $\eta_i(\tau) = \eta_i(0)[1 \pm \Delta v k A \tau]$ with the + (−) sign associated with $i=1$ (2), as expected.

Equations (10)–(13) are valid for incompressible fluids, arbitrary thickness t, and, as with all formulas in this paper, for small perturbations only. The corresponding finite-thickness *compressible* problem may well be intractable analytically, given the complexities of the classical configuration, as discussed above. Alternatively, one may carry out numerical simulations which include compressibility and nonlinearity. For lack of space our simulations will be reported elsewhere.

The point we wish to make is that because the SF_6 layer in the experiments was relatively thin and, more importantly, the perturbations were nonlinear to start with, we cannot use the classical expression, Eq. (3), to infer growth rates. On the other hand, the experiments did have the great advantage of using no membranes. There have been other membraneless experiments [10], but this time a continuous density gradient at the interface, though interesting by itself, spoils the comparison with classical formulas. It is somewhat ironic that after so much work the classical RM experiment (two semi-infinite fluids, no membrane, small-amplitude perturbation) remains to be done. As we discussed above, even the theory is quite challenging, as evidenced by Fraley's analysis. Few people, including us, expected so much complexity for an apparently "simple, linear" problem.

This is not to say that finite-thickness experiments, even in the linear regime, are not interesting. Our final remark concerns Eq. (12): If the initial conditions are such that $\eta_1(0)/\eta_2(0) = \sin\theta$ then $\eta_1(\tau) = \eta_1(0)$. In other words $\dot\eta_1 = 0$ and the perturbation at the first interface freezes-out. To freeze η_2 one must arrange $\eta_2(0)/\eta_1(0) = \sin\theta$. We label this phenomenon *double-interface freeze-out* because a second interface is necessary here to influence the first interface. Like double-shock freeze-out, but quite unlike single-shock freeze-out, we find that double-interface freeze-out occurs for incompressible as well as compressible fluids. Examples will be given elsewhere.

This work was performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract No. W-7405-ENG-48.

[1] R. D. Richtmyer, Commun. Pure Appl. Math. **13**, 297 (1960).
[2] E. E. Meshkov, Fluid Dynamics **4**, 101 (1969).
[3] Collections of recent work include *Advances in Compressible Turbulent Mixing*, edited by W. P. Dannevik, A. C. Buckingham, and C. E. Leith (Lawrence Livermore Laboratory Report No. CONF-8810234, 1992); *Proceedings of the 3rd International Workshop on the Physics of Compressible Turbulent Mixing, Abbey of Royaumont, France, 1992*; Proceedings of the 4th International Workshop on the Physics of Compressible Turbulent Mixing, Cambridge, England, 1993 (to be published).
[4] K. A. Meyer and P. J. Blewett, Phys. Fluids **15**, 753 (1972).
[5] R. F. Benjamin, in *Advances In Compressible Turbulent Mixing* (Ref. [3]), p. 341.
[6] K. O. Mikaelian, Phys. Rev. A **31**, 410 (1985).
[7] K. O. Mikaelian, Phys. Fluids (to be published).
[8] G. Fraley, Phys. Fluids **29**, 376 (1986).
[9] J. W. Jacobs, D. L. Klein, D. G. Jenkins, and R. F. Benjamin, Phys. Rev. Lett. **70**, 583 (1993).
[10] M. Brouillette and B. Sturtevant, Physica (Amsterdam) **37D**, 248 (1989).