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Squeezed States for General Systems

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We propose a ladder-operator method for obtaining the squeezed states of general symmetry systems. It is a generalization of the annihilation-operator technique for obtaining the coherent states of symmetry systems. We connect this method with the minimum-uncertainty method for obtaining the squeezed and coherent states of general potential systems, and comment on the distinctions between these two methods and the displacement-operator method.

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Coherent states are important in many fields of theoretical and experimental physics [1,2]. Similarly, the generalization of coherent states, squeezed states, has become of more and more interest in recent times [3,4]. This is especially true in the fields of quantum optics [5] and gravitational wave detection [6].

However, one limitation is that, with the exception we describe below, essentially all work on squeezed states has concentrated on the harmonic oscillator system. In this Letter we describe a generalization of squeezed states to arbitrary symmetry systems, and its relationship to squeezed states obtained for general potentials.

We begin by reviewing coherent states and squeezed states.

(1) Displacement-operator method.—For the harmonic oscillator, coherent states are described by the unitary displacement operator acting on the ground state [7,8]:

$$\begin{aligned} D(\alpha)|0\rangle &= \exp[\alpha a^\dagger - \alpha^* a]|0\rangle \\ &= \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv |\alpha\rangle. \end{aligned} \tag{1}$$

The generalization of this method to arbitrary Lie groups has a long history [1,2,7,9]. One simply applies the displacement operator, which is the unitary exponentiation of the factor algebra, onto an extremal state.

As to squeezed states, this method has basically only been applied to harmonic oscillatorlike systems [3,4]. One applies the SU(1,1) displacement operator onto the coherent state,

$$D(\alpha)S(z)|0\rangle = |(\alpha, z)\rangle, \qquad S(z) = \exp[zK_+ - z^*K_-], \tag{2}$$

where K_+, K_-, and K_0 form an su(1,1) algebra among themselves:

$$K_+ = \tfrac{1}{2}a^\dagger a^\dagger, \quad K_- = \tfrac{1}{2}aa, \quad K_0 = \tfrac{1}{2}(a^\dagger a + \tfrac{1}{2}), \tag{3}$$

$$[K_0, K_\pm] = \pm K_\pm\ , \quad [K_+, K_-] = -2K_0. \tag{4}$$

The ordering of DS vs SD in Eq. (2) is unitarily equivalent, amounting to a change of parameters. (Supersymmetric extensions of the above exist [10].)

(2) Ladder- (annihilation-) operator method.—For the harmonic oscillator, the coherent states are the eigenstates of the destruction operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

This follows from Eq. (1), since $0 = D(\alpha)a|0\rangle = (a - \alpha)|\alpha\rangle$. These states are the same as the displacement-operator coherent states. The generalization to arbitrary Lie groups is straightforward, and has also been widely studied [1,2].

(3) Minimum-uncertainty method.—This method, which intuitively harks back to Schrödinger's discovery

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of the coherent states [11], has been applied to general Hamiltonian potential systems, to obtain both generalized coherent states and generalized squeezed states [12,13]. One starts with the classical problem and transforms it into the "natural classical variables," X_c and P_c, which vary as the sin and the cos of the classical ωt. The Hamiltonian is therefore of the form $P_c^2 + X_c^2$. One then takes these natural classical variables and transforms them into "natural quantum operators." Since these are quantum operators, they have a commutation relation and uncertainty relation:

$$[X, P] = iG, \quad (\Delta X)^2(\Delta P)^2 \geq \tfrac{1}{4}\langle G\rangle^2. \tag{6}$$

The states that minimize this uncertainty relation are given by the solutions to the equation

$$\begin{aligned} Y\psi_{ss} \equiv & \left(X + \frac{i\langle G\rangle}{2(\Delta P)^2}P\right)\psi_{ss} \\ = & \left(\langle X\rangle + \frac{i\langle G\rangle}{2(\Delta P)^2}\langle P\rangle\right)\psi_{ss}. \end{aligned} \tag{7}$$

Note that of the four parameters $\langle X\rangle, \langle P\rangle, \langle P^2\rangle$, and $\langle G\rangle$, only three are independent because they satisfy the equality in the uncertainty relation. Therefore,

$$(X + iBP)\,\psi_{ss} = C\psi_{ss}, \quad B = \frac{\Delta X}{\Delta P}, \quad C = \langle X\rangle + iB\langle P\rangle. \tag{8}$$

Here B is real and C is complex. These states, $\psi_{ss}(B, C)$, are the minimum-uncertainty states for general potentials [12,13]. Using later parlance, they are the squeezed states for general potentials [4]. Then B can be adjusted to B_0 so that the ground eigenstate of the potential is a member of the set. Then these restricted states, $\psi_{ss}(B = B_0, C) = \psi_{cs}(B_0, C)$, are the minimum-uncertainty coherent states for general potentials.

It can be intuitively understood that $\psi_{ss}(B, C)$ and $\psi_{ss}(B_0, C)$ are the squeezed and coherent states by recalling the situation for the harmonic oscillator. The coherent states are the displaced ground state. The squeezed states are Gaussians that have different widths than that of the ground state Gaussian, which are then displaced.

New ladder-operator method for general squeezed states.—General annihilation-operator (or ladder-operator) coherent states are the eigenstates of the lowering operator (given a lowest extremal state). We now propose a generalization to squeezed states, including those for arbitrary symmetry systems: The general ladder-operator squeezed states are the eigenstates of a linear combination of the lowering and raising operators. (See comment III, below, concerning previous special cases.)

We will show how the minimum-uncertainty method for obtaining generalized squeezed states can be used as an intuitive tool to aid in understanding the ladder-operator method for obtaining generalized squeezed states. We will do this with two specific examples. Once that is done, the ladder-operator method can be applied to general symmetry systems, independent of whether they come from a Hamiltonian system in the manner of the minimum-uncertainty method above. Such is our third example.

Example I.—First we reexamine the harmonic oscillator, starting from the minimum-uncertainty method. Here X and P are obviously x and p. (We use dimensionless units.) Then we have

$$Y = x + s^2\frac{d}{dx}, \tag{9}$$

where we have presciently labeled B as s^2. (For the limit to coherent states, it turns out that $B = 1$.)

Now writing x and p in terms of creation and annihilation operators, $x = (a + a^\dagger)/\sqrt{2}$, $p = (a - a^\dagger)/i\sqrt{2}$, we find

$$\begin{aligned} \sqrt{2}\left[a\left(\frac{1+s^2}{2}\right) + a^\dagger\left(\frac{1-s^2}{2}\right)\right]\psi_{ss}(s^2, x_0 + is^2p_0) \\ = [x_0 + is^2p_0]\psi_{ss}(s^2, x_0 + is^2p_0). \end{aligned} \tag{10}$$

Therefore, the squeezed states are eigenstates of a linear combination of the annihilation and creation operators. Specifically, these states are

$$\psi_{ss}(x) = [\pi s^2]^{-1/4}\exp\left[-\frac{(x - x_0)^2}{2s^2} + ip_0x\right]. \tag{11}$$

The relationships to the displacement-operator parameters are $z = re^{i\phi}$, $r = \ln s$, $\sqrt{2}\mathcal{R}(\alpha) = x_0$, and $\sqrt{2}\mathcal{I}(\alpha) = p_0$. (The phase, ϕ, is an initial time displacement.)

We note, with hindsight, that the success of this method will not be totally surprising. In many exactly solvable potential systems, the natural quantum operators of the minimum-uncertainty method were found to be Hermitian combinations of the n-dependent raising and lowering operators [12,13]. Here, however, one must generalize to full operators: $n \to n(H)$. Furthermore, in other harmonic-oscillator-like systems, with a Bogoliubov transformation, this method applies. (See below.)

Example II.—We demonstrate this method with the symmetry of the harmonic oscillator with centripetal barrier. Previously, the coherent states for this particular example were found with the minimum-uncertainty method, but not the squeezed states [13]. Therefore, it is an ideal system since, at the end, we can connect to the coherent states obtained from the minimum-uncertainty method.

This system contains an su(1,1) algebra [14]. Its elements are

$$L_\pm = \frac{1}{4\nu}\frac{d^2}{dz^2} \mp \frac{1}{2}z\frac{d}{dz} \mp \frac{1}{4} + \frac{\nu}{4}z^2 - \frac{\nu}{4z^2}, \tag{12}$$

$$L_0 = \frac{H}{4\nu} + \frac{\nu}{2}, \quad H = -\frac{d^2}{dz^2} + \nu^2\left(\frac{1}{z} - z\right)^2. \tag{13}$$

In terms of the X and P minimum-uncertainty operators [13], we find

$$X = \frac{L_- + L_+}{\nu} = z^2 - \left(1 + \frac{H}{2\nu^2}\right),$$
$$P = \frac{2(L_- - L_+)}{i} = \frac{1}{i}\left[2z\frac{d}{dz} + 1\right]. \tag{14}$$

Therefore, the squeezed states for this system are formed by the solution to the equation

$$0 = \left[y\frac{d^2}{dy^2} + \left(\frac{1}{2} + 2\nu B y\right)\frac{d}{dy} + \frac{1}{4}\left(y - \frac{\nu^2}{y} + 2B\nu\right) - \frac{\nu C}{2}\right]\psi_{ss}, \tag{15}$$

where we have changed variables to $y = \nu z^2$. The squeezed state solutions to this equation are

$$\psi_{ss} = N \exp[-y(\nu B + \gamma)][y^{\lambda+\frac{1}{2}}] \times \Phi\left(\left[\frac{\nu C}{4\gamma} + \frac{1}{2}\left(\lambda + \frac{3}{2}\right)\right],\ \left[\lambda + \frac{3}{2}\right];\ 2\gamma y\right), \tag{16}$$

where $\Phi(a, b; c)$ is the confluent hypergeometric function $\sum_{n=0}^{\infty} \frac{(a)_n c^n}{(b)_n\, n!}$, $\gamma = \sqrt{\nu^2 B^2 - \frac{1}{4}}$, and $\lambda(\lambda+1) = \nu^2$. In the limit where $B \to 1/2\nu$, these become the coherent states given in Ref. [13],

$$\psi_{cs} = \left[\frac{2\nu^{1/2} e^{-\nu \mathcal{R}(C)}}{I_{\lambda+1/2}(\nu|C|)}\right]^{1/2} e^{-y/2} y^{1/4} I_{\lambda+1/2}[(2\nu C y)^{1/2}], \tag{17}$$

where I is the modified Bessel function.

Example III.—We now consider a symmetry system which does not have as its origin a Hamiltonian system. We consider the su(1,1) symmetry of Eqs. (3) and (4). Our ladder-operator squeezed states are thus the solutions to

$$\left[\left(\frac{1+s}{2}\right) aa + \left(\frac{1-s}{2}\right) a^\dagger a^\dagger\right]\psi_{ss} = \beta^2 \psi_{ss}, \tag{18}$$

where the analog of B is s and the role of C is taken by β^2. Using the differential representations of the ladder operators, Eq. (18) can be written in the form

$$\left[\frac{d^2}{dy^2} + 2ys\frac{d}{dy} + y^2 + (s - 2\beta^2)\right]\psi_{ss} = 0. \tag{19}$$

Observe that the ladder operators raise and lower the number states by two units. Therefore, there will be two solutions to this equation, one containing only even number states and one containing only odd number states. We will designate these as ψ_{Ess} and ψ_{Oss}. These solutions are

$$\psi_{Ess} = N_E \exp\left[-\frac{-y^2}{2}(s + \sqrt{s^2-1})\right] \times \Phi\left(\left[\frac{1}{4} + \frac{\beta^2}{2\sqrt{s^2-1}}\right],\ \frac{1}{2};\ y^2\sqrt{s^2-1}\right), \tag{20}$$

$$\psi_{Oss} = N_O\, y \exp\left[-\frac{-y^2}{2}(s + \sqrt{s^2-1})\right] \times \Phi\left(\left[\frac{3}{4} + \frac{\beta^2}{2\sqrt{s^2-1}}\right],\ \frac{3}{2};\ y^2\sqrt{s^2-1}\right). \tag{21}$$

In the limit $s \to 1$, these become the even and odd coherent states:

$$\psi_{Ecs} = \left[\frac{e^{-\beta^2}}{\pi^{1/2}\cosh|\beta|^2}\right]^{1/2} \exp\left[-\frac{1}{2}y^2\right]\cosh(\sqrt{2}\beta y), \tag{22}$$

$$\psi_{Ocs} = \left[\frac{e^{-\beta^2}}{\pi^{1/2}\sinh|\beta|^2}\right]^{1/2} \exp\left[-\frac{1}{2}y^2\right]\sinh(\sqrt{2}\beta y). \tag{23}$$

Using generating formulas, these can be written in the number state basis as

$$\psi_{Ecs} = [\cosh|\beta|^2]^{-1/2}\sum_{n=0}^{\infty}\frac{\beta^{2n}}{\sqrt{(2n)!}}|2n\rangle, \tag{24}$$

$$\psi_{Ocs} = [\sinh|\beta|^2]^{-1/2}\sum_{n=0}^{\infty}\frac{\beta^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle. \tag{25}$$

Up to the normalization, these are the "even and odd coherent states" previously found in Ref. [15]. Although this system did not come from a Hamiltonian, one could have used a minimum-uncertainty principle to obtain the same states by starting with the commutation relation $[K_+, K_-] = -2K_0$. However, one does not obtain the same coherent states from the displacement-operator method. Those coherent states, defined by $S(z)|0\rangle$, are the squeezed-vacuum Gaussian of Eq. (11) with $x_0 = p_0 = 0$.

Comment I.—The above discussion brings us to the displacement-operator method. Although it is the natural method for defining coherent states for Lie algebras, there is as yet no well-known general extension of this method to define general displacement-operator squeezed states. This has been touched upon in discussions [16] about higher-order generalizations of the "squeeze operator," $S(z)$. In particular, although harmonic-oscillator-like systems admit squeeze operators (or Bogoliubov transformations) connecting the displacement-operator and ladder-operator methods [17,18], the appropriate generalization of these squeeze operators has not been found. Therefore, for now, the ladder-operator method is generally connected only to the minimum-uncertainty method.

Comment II.—In this vein, for finite-dimensional representations, such as for angular momentum coherent states, the ladder-operator method does not allow a solution for coherent states, although the displacement-operator method does [17]. Contrariwise, for squeezed states, we observe that the opposite is true.

Comment III.—The above three examples have all been cases where $A_- = (A_+)^\dagger$. Sometimes that is not the case, as in certain potential systems whose eigenenergies are not equally spaced [12,13]. Then, as in Eq. (13), one should use the operator form for "n": $A_n \to A_{n(H)}$, to connect to the minimum-uncertainty method. In these cases, the ladder-operator coherent and squeezed states can be different from, though related to, their minimum-uncertainty counterparts.

An application of these ideas to Rydberg wave packets will appear elsewhere [19]. It is observed that, in general, these packets are squeezed states.

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