Exciton, Spinon, and Spin Wave Modes in a Soluble One-Dimensional Many-Body System

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(Received 20 April 1993)

In this paper, we present the exact solution (in the thermodynamic limit) to a one-dimensional, two-component, quantum many-body system in which like particles interact with a pair potential $s(s+1)/\sinh^2(r)$, while unlike particles interact with a pair potential $-s(s+1)/\cosh^2(r)$. We first give a proof of integrability, then derive the coupled equations determining the complete spectrum. All singularities occur in the ground state when there are equal numbers of the two components; we give explicit results for the ground state and low-lying states in this case.

PACS numbers: 71.45.Gm, 75.30.Ds

We present the exact solution (in the thermodynamic limit) to a one-dimensional, two-component, quantum many-body system of considerable complexity. The two kinds of particles are distinguished by a quantum number $\sigma = \pm 1$, which may be thought of as either spin or charge. The system is defined by a Hamiltonian with pair potential

$$v_{jk}(x) = s(s+1)\left[\frac{1+\sigma_j\sigma_k}{2\sinh^2(x)} - \frac{1-\sigma_j\sigma_k}{2\cosh^2(x)}\right] . \quad (1)$$

We assume $s \geq -1$. Thus for $s > 0$, like particles repel, while unlike particles attract. When like particles are near, the repulsive potential increases as $1/r^2$, while for large separations, both potentials decay exponentially with a decay length we take as our length scale, and hence unity.

This system was first introduced by Calogero [1], who showed it to be integrable. Sutherland [2] soon afterward showed that the system could be exactly solved, and gave the solution for a single component system. Our present solution for the two-component system exploits in a fundamental way the integrability of the system [3], so we first discuss this point.

For a classical system of N one-dimensional particles, Lax [4], Moser [5], and Calogero [6] have shown that for certain potentials one can find two Hermitian $N \times N$ matrices L and A that obey the Lax equation $dL/dt = i[A, L]$ and $\det[L - \omega\mathbb{1}]$ is a constant of motion. Expanding the determinant in powers of ω, we find N integrals of motion in involution, and thus the system is integrable. Calogero showed that the quantum system is also completely integrable.

For systems which support scattering, in the distant past and future the Lax matrix L approaches a diagonal matrix with the momenta as diagonal elements, so that $\det[L - \omega\mathbb{1}] = \prod_{1\leq j\leq N}(p_j - \omega)$. Thus the individual momenta p_j are conserved in a collision, and hence, as emphasized by Sutherland [2], the wave function is given asymptotically by Bethe's ansatz.

The proof of Calogero, however, is very difficult, and only briefly sketched in the literature. For that reason, we now offer an alternative proof of integrability based on a method of Shastry [7]. Let us write the Lax matrices as

$$A_{jk} = \delta_{jk}\sum_{l\neq j}\alpha'_{jl} + (\delta_{jk} - 1)\alpha'_{jk}, \quad (2)$$

$$L_{jk} = \delta_{jk}p_j + i(1-\delta_{jk})\alpha_{jk}, \quad (3)$$

where

$$\alpha_{jk} = -s\left[\frac{1+\sigma_j\sigma_k}{2}\coth(x_j - x_k) + \frac{1-\sigma_j\sigma_k}{2}\tanh(x_j - x_k)\right], \quad (4)$$

and $\alpha'_{jk} = \partial\alpha_{jk}/\partial x_j$. Then if the two-body potential v_{jk} is given by $v_{jk} = \alpha_{jk}^2 + \alpha'_{jk} - s^2$, we find that the quantum Lax equation $[H, L_{jk}] = \sum_{1\leq l\leq N}(L_{jl}A_{lk} - A_{jl}L_{lk})$ is satisfied. This potential, however, is exactly the potential for our system.

We now observe that the Lax A matrix has the following very important property. Defining a vector ζ with $\zeta_j = 1$, we see $A\zeta = \zeta^\dagger A = 0$. This allows us to construct constants of motion by $I_n = \zeta^\dagger L^n \zeta$, since

$$[H, I_n] = \zeta^\dagger[H, L^n]\zeta \, , \quad (5)$$

$$= \zeta^\dagger \sum_{0<j<n-1}\left\{L^j[H,L]L^{n-1-j}\right\}\zeta \, , \quad (6)$$

$$= \zeta^\dagger \sum_{0<j<n-1}\left\{L^j[A,L]L^{n-1-j}\right\}\zeta \, , \quad (7)$$

$$= \zeta^\dagger\left\{AL^{n-1} - L^{n-1}A\right\}\zeta = 0 \, . \quad (8)$$

By Jacobi's relation for commutators, $[I_n, I_m]$ is a constant of motion, and since this is a system that supports scattering, we see $[I_n, I_m] \to 0$, and hence the system is completely integrable.

Having shown the system to be integrable, we then know the asymptotic wave function to be of the Bethe ansatz form, and the only input needed for the Bethe ansatz is the solution to the two-body problem. We summarize the results in the center of mass frame [8] below.

For like particles, the potential is $s(s+1)/\sinh^2(r)$, and the wave function is given asymptotically as

$$\Psi(r) = \begin{cases} r^{s+1}, & r \to 0+, \\ e^{-ikr} + S(2k)e^{ikr}, & r \to +\infty. \end{cases} \quad (9)$$

The scattering amplitude $S(k)$ is given by

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$$S(k) = -\frac{\Gamma(1+ik/2)\Gamma(1+s-ik/2)}{\Gamma(1-ik/2)\Gamma(1+s+ik/2)} . \quad (10)$$

For bosons (fermions) the wave function must be (anti)symmetric, so the scattering amplitude for transmission will be $\pm S(k)$. In what follows, we will drop factors of -1 in the scattering amplitudes, assuming that they are taken care of by either the choice of statistics of the particles, the choice of quantum numbers as half-odd integers, or the choice of number of particles as even or odd.

For unlike particles, with the potential $-s(s+1)/\cosh^2(r)$, the wave function is given asymptotically as

$$\Psi(r) = \begin{cases} e^{ikr} + R(2k)e^{-ikr}, & r \to -\infty, \\ T(2k)e^{ikr}, & r \to +\infty. \end{cases} \quad (11)$$

The reflection and transmission amplitudes are $R(k) = S(k)r(k)$, $T(k) = S(k)t(k)$, where

$$r(k) = \frac{\sin \pi s}{\sin \pi(s+ik/2)}, \quad (12)$$

$$t(k) = -\frac{\sin \pi ik/2}{\sin \pi(s+ik/2)}. \quad (13)$$

There are bound states, labeled by an index $m = 1, 2, \ldots, M$, according to increasing energy, with parity $(-1)^{m-1}$. Bound states appear as poles of the reflection and transmission amplitudes, $R(k_1 - k_2)$ and $T(k_1 - k_2)$ on the positive imaginary axis, given by $k_{1,2} = k \pm i\kappa, k > 0$. The momentum and energy of such a bound state is $P = 2k$ and $E = k^2 - \kappa^2$. From the particular form of the reflection and transmission amplitudes, we find $\kappa_m = s + 1 - m$, where $m = 1, 2, \ldots, M$, and $M(s)$ is the smallest integer larger than s. There are no bound states for $-1 \leq s \leq 0$. Threshhold values of s are $s = 0, 1, 2, \ldots$, and at these values, the reflection amplitude vanishes. At the bound state poles we also find $r(2i\kappa_m)/t(2i\kappa_m) = (-1)^{m-1}$. We call the bound states pairs.

We know the Yang-Baxter equations must hold, and we can verify this explicitly. For a two-component system the Yang-Baxter equations are equivalent to $r_2 = r_3 r_1 + t_3 r_2 t_1$ and $r_3 t_2 = r_3 t_1 + t_3 r_2 r_1$, where $r_1 = r(k_1 - k_2), r_2 = r(k_1 - k_3), r_3 = r(k_2 - k_3)$, etc., for t_j. A degenerate situation occurs at a pole in r_3 and t_3, when $k_2 - k_3 = 2i\kappa_m$. There, since $r_3/t_3 = (-1)^{m-1}$, the equations become $0 = r_1 + (-1)^{m-1} r_2 t_1$ and $t_2 = t_1 + (-1)^{m-1} r_2 r_1$, where $r_{2,1} = r(k \pm i\kappa_m)$, etc., for t_j. These relationships will be important when we calculate phase shifts.

If a particle of type m passes through a particle of type m', without reflection, then we have a scattering amplitude $\exp[-i\theta_{mm'}(k_1 - k_2)]$, and a corresponding phase shift $\theta_{mm'}(k)$. Let us label the unbound particle by $m = 0$. Then we have found

$$\theta_{00}(k) = i \ln \left[\frac{\Gamma(1+ik/2)\Gamma(1+s-ik/2)}{\Gamma(1-ik/2)\Gamma(1+s+ik/2)} \right] . \quad (14)$$

In general $\theta_{mm'}(k) = -\theta_{mm'}(-k) = \theta_{m'm}(k)$ and $\theta_{mm'}(0) = 0$.

Now, consider the scattering of a particle k_1 on a pair of two particles with momenta $k_2 \pm i\kappa_m$. Let $k = k_1 - k_2$. Then using the degenerate Yang-Baxter equations, we find for the scattering amplitude

$$S(k+i\kappa_m)S(k-i\kappa_m)t(k+i\kappa_m) = \exp[-i\theta_{0m}(k)]. \quad (15)$$

Using the explicit forms, we can verify that $\theta_{0m}(k)$ is real for k real.

Finally, we view the scattering of a pair from a pair as the scattering of two particles with momenta $k_1 \pm i\kappa_m$ from a pair with $k_2 \pm i\kappa_{m'}$. This gives us a net phase shift $\theta_{mm'}(k) = \theta_{0m'}(k - i\kappa_m) + \theta_{0m'}(k + i\kappa_m)$. Again, using the explicit forms, we can verify that $\theta_{mm'}(k)$ is real for k real, and symmetric in m, m'.

To summarize, we have $N_{\uparrow,\downarrow}$ particles with $\sigma = \pm 1$, and $N = N_\uparrow + N_\downarrow$, $N_\uparrow \geq N_\downarrow$. Let there be N_m bound states of each type. Then the number of unbound particles is $N_0 = N - 2\sum_{1\leq m\leq M} N_m$; they would correspond to spinons (ions) in the spin (charge) picture. Of these particles, we have N_{-1} with spin down; let us call them spin waves. Clearly $N_{-1} = N_\downarrow - \sum_{1\leq m\leq M} N_m$ and $N_{-1} \leq N_0/2$.

We still must treat the dynamics of the spin waves; since they are not "real" particles, but only correlations in the quantum numbers of particles, they carry no momentum or energy. Thus, defining $\eta_m = 0$ for $m = -1$, 1 for $m = 0$, and 2 for $m = 1, 2, \ldots, M(s)$, we can write the momentum and energy as

$$P = \sum_{-1\leq m\leq M} \eta_m \sum_{k_m} k_m, \quad (16)$$

$$E = \frac{1}{2} \sum_{-1\leq m\leq M} \eta_m \sum_{k_m} k_m^2 - \sum_{1\leq m\leq M} N_m \kappa_m^2 . \quad (17)$$

Pairs pass through particles and pairs with only a phase shift and no reflection, but particles scatter from particles with reflection. We write the asymptotic wave function explicitly in the Bethe ansatz form, and for now consider only the N_0 particles. We use the spin language, so $\sigma_z(j) = \pm 1$ according to whether the jth particle in the ordering $x_1 < \cdots < x_{N_0}$ has spin up or down. A choice for all $\sigma_z(j)$ we denote simply by σ. Then asymptotically the wave function is given by

$$\Psi(x|\sigma) \to \sum_P A(P|\sigma) \exp \left[i \sum_{1\leq j\leq N_0} x_j k_{Pj} \right] . \quad (18)$$

The summation is over all the $N_0!$ permutations of the momenta. We arrange the $A(P|\sigma)$ for fixed P as a column vector $\xi(P)$. Then the Yang-Baxter equations ensure that we can find a consistent set of amplitudes $A(P|\sigma)$, by finding the simultaneous eigenvector of the N_0 equations

$$e^{ik_jL}\prod_{1\le n\le N_0}S(k_j-k_n)X_{j,j-1}\cdots X_{j,1}X_{j,N_0}\cdots X_{j,j+1}\xi(I)=\xi(I)\ . \tag{19}$$

In this equation, the $X_{j,n}$ are operators given as

$$X_{j,n}=\frac{1+t(k)}{2}\mathbb{1}+\frac{1-t(k)}{2}\sigma_z(j)\sigma_z(n) +\frac{r(k)}{2}\left[\sigma_x(j)\sigma_x(n)+\sigma_y(j)\sigma_y(n)\right], \tag{20}$$

where $k=k_{P_j}-k_{P_n}$.

These eigenvalue equations can again be solved by a Bethe ansatz for the N_{-1} overturned spins—the spin waves—on a lattice of N_0 particles: either (i) directly, by the methods of Yang [9]; (ii) with commuting transfer matrices, by the methods of Baxter [10]; or (iii) by quantum inverse scattering methods of Faddeev and Takhtajan [11]. We are not aware that these equations have appeared before in the solution of a quantum many-body problem, although the low-density case has often appeared, for instance first in Yang's original solution for δ-function fermions.

The solution is sufficiently technical that we postpone discussion to a later publication. However, the result has many interesting physical consequences, and that is what we want to discuss in this Letter. One finds for the eigenvalues of the previous equations

$$e^{ik_jL}\prod_{1\le n\le N_0}S(k_j-k_n)\prod_{1\le q\le N_{-1}}\frac{\sin\pi[s-i(k_j-\lambda_q)]/2}{\sin\pi[s+i(k_j-\lambda_q)]/2}=1\ . \tag{21}$$

In this equation, the λ's are the momenta of the spin waves, and are determined from the equation

$$\prod_{1\le q\le N_{-1}}\frac{\sin\pi[s+i(\lambda_j-\lambda_q)/2]}{\sin\pi[s-i(\lambda_j-\lambda_q)/2]}\prod_{1\le n\le N_0}\frac{\sin\pi[s-i(\lambda_j-k_n)]/2}{\sin\pi[s+i(\lambda_j-k_n)]/2}=1\ . \tag{22}$$

We now have our two final phase shifts, for particle–spin-wave and spin-wave–spin-wave scattering:

$$\theta_{0,-1}(k)=i\ln\left[\frac{\sin\pi[s-ik]/2}{\sin\pi[s+ik]/2}\right], \tag{23}$$

$$\theta_{-1,-1}(k)=i\ln\left[\frac{\sin\pi[s+ik/2]}{\sin\pi[s-ik/2]}\right]. \tag{24}$$

As noted, there is no phase shift for spin-wave-pair scattering. contribute no energy or momentum directly.

Let us now impose periodic boundary conditions and take any particle, pair, or spin wave around a ring of large circumference. Along the way, it suffers a phase change as it scatters from every other particle, pair, or spin wave, plus a phase change of PL, where $P=\eta k$ is its own momentum. Periodicity requires that this phase change be an integer multiple of 2π, the integer being the quantum number. We write this statement as coupled equations in a rather symbolic form:

$$L\eta_mk_m=2\pi I_m(k_m)+\sum_{-1\le m'\le M}\sum_{k'_{m'}}\theta_{m,m'}(k_m-k_{m'})\ , \quad m=-1,0,1,\ldots,M\ . \tag{25}$$

Here the $I_m(k_m)$ are the quantum numbers, the only subtlety being that for the spin waves, I_{-1} ranges only over $1,\ldots,N_0$.

In this Letter, we give explicit results for the ground state and low-lying states when $N_\uparrow=N_\downarrow$, which we call the spin or charge zero sector. This certainly is the most interesting case, since all singularities in the $(N_\uparrow,N_\downarrow)$ ground state phase diagram occur for $N_\uparrow=N_\downarrow$. In fact, as we shall see, for $s>0$, the chemical potential has a discontinuity across the line $N_\uparrow=N_\downarrow$, and thus the system is an antiferromagnet (insulator), although not of the Néel (Mott) type. For $-1<s<0$, there is a weak singularity at $N_\uparrow=N_\downarrow$, without a discontinuity in the chemical potential.

For $s>0$ the ground state consists of a spin fluid of type $m=1$, and thus spin 0. This is the bound state with lowest binding energy, when $\kappa=s$, and so $P(k)=2k$ and $E(k)=k^2-s^2$. In the ground state, the k's for the pairs distribute themselves densely with a density $\rho(k)$, between limits $\pm B$, normalized so that

$$N_1/L=\int_{-B}^{B}\rho(k)dk=N/2L. \tag{26}$$

The energy and momentum are given by

$$P/L=2\int_{-B}^{B}\rho(k)k\,dk=0, \tag{27}$$

$$E/L=\int_{-B}^{B}\rho(k)k^2\,dk-s^2N_1/L. \tag{28}$$

The integral equation which determines $\rho(k)$ is

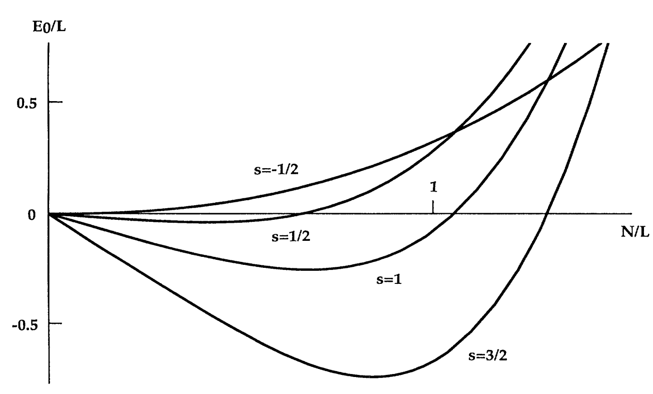

FIG. 1. Ground state energy per unit length E_0/L versus density N/L for $s=-1/2,1/2,1,3/2$.

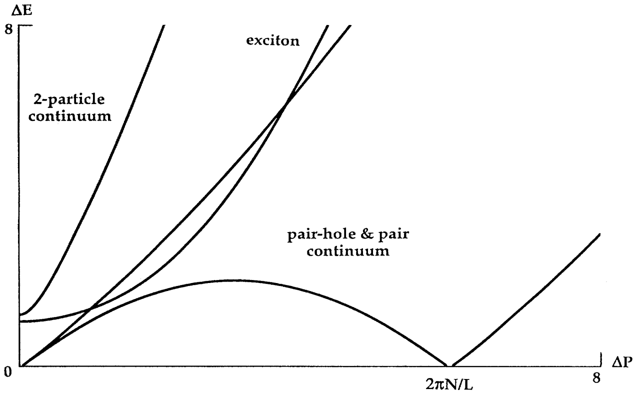

FIG. 2. Energy above the ground state energy versus momentum (dispersion relations) for the low-lying excitations when $s = 3/2$ and density $N/L = 0.943$.

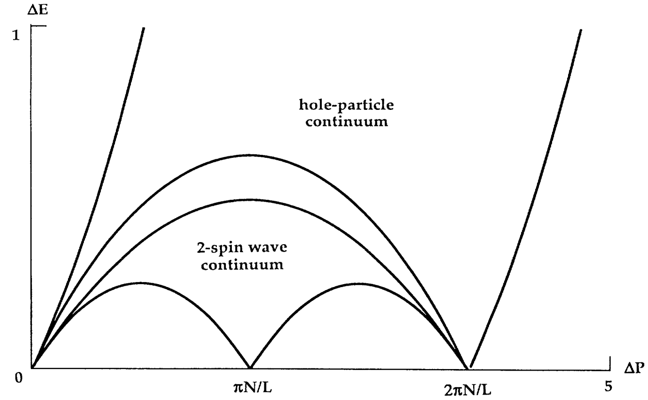

FIG. 3. Energy above the ground state energy versus momentum (dispersion relations) for the low-lying excitations when $s = -1/2$ and density $N/L = 0.600$.

$$1/\pi = \rho(k) + \frac{1}{2\pi}\int_{-B}^{B} \theta'_{11}(k-k')\rho(k')dk'. \quad (29)$$

In Fig. 1 we show E_0/L versus N/L for selected values of $s = 1/2, 1, 3/2$.

Having determined the ground state properties of the system, we now determine the low-energy excited states. They are given by the following: (i) Remove a pair from the ground state distribution, and place it outside the limits; we call this creating a pair hole and a pair, and it gives a two parameter continuum. (ii) Break a pair to give two particles, one spin up and the other spin down; this also gives a two parameter continuum. (iii) Excite a pair into a higher energy bound state, if allowed; these we call excitons, and they have single parameter dispersion relations. (Away from the zero sector, we can have in addition spin waves. These will be important for $s < 0$.)

By the techniques of Yang and Yang [12], the dispersion relations are given parametrically by

$$\Delta P = \sum_m \left[\eta_m k_m - \int_{-B}^{B} \theta_{m1}(k_m - k)\rho(k)dk \right], \quad (30)$$

$$\Delta E = \sum_m \left[\frac{\eta_m k_m^2}{2} - \frac{1}{2\pi}\int_{-B}^{B} \theta'_{m1}(k_m - k)\epsilon(k)dk \right]. \quad (31)$$

Here $\epsilon(k)$ is the solution to the integral equation

$$k^2 - s^2 - \mu_1 = \epsilon(k) + \frac{1}{2\pi}\int_{-B}^{B} \theta'_{11}(k-k')\epsilon(k')dk'. \quad (32)$$

The chemical potential μ_1 is the chemical potential for pairs, given by $\partial E_0/\partial N_1$. The results are shown in Fig. 2 for $s = 3/2$, $B = 3/2$, $d = N/L = 0.943$, $E_0/L = -0.691$, and $\mu_1 = 1.215$. The gap for the creation of two particles is $\Delta E = 1.170$, and is equal to the discontinuity of the chemical potential across the line $N_\uparrow = N_\downarrow$. The exciton with $m = 2$ is the only exciton allowed at this value of s, and has a gap of $\Delta E = 1.017$.

For $0 > s > -1$, in the zero sector, we have two coupled equations for N particles and $N/2$ spin waves. However, in the zero sector, the limits of the spin-wave distribution are $\pm\infty$. Thus we can solve by Fourier transforms for the spin-wave distribution in terms of the particle distribution, and then substitute this into the particle equation, giving us a single integral equation for the distribution of particles $\rho(k)$. In Fig. 1, we show E_0/L versus N/L for $s = -1/2$. The excited states in the zero sector are given by the following: (i) Remove a particle from the ground state distribution, and place it outside the limits; we call this creating a hole and a particle, and it gives a two parameter continuum. (ii) Remove a spin wave from the ground state distribution, and place it on the line with imaginary part equal to i; we call this creating two spin waves, one with spin up and the other with spin down. It gives a two parameter continuum, familiar from the Heisenberg-Ising model. The results are shown in Fig. 3, for $s = -1/2$, $B = 1$, $d = N/L = 0.600$, $E_0/L = 0.094$, and $\mu = 0.374$.

Finally, we remark that all thermodynamics can be explicitly calculated, since there are no ambiguities with counting states or difficulties with strings of length greater than 2.

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