Novel Hierarchy of the SU(N) Electron Models and Edge States of Fractional Quantum Hall Effect

Norio Kawakami
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan
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A novel hierarchy of the one-dimensional SU(N) electron models with $1/r^2$ interaction is proposed and solved by the asymptotic Bethe ansatz both for the continuum and lattice cases. The construction of the hierarchy is closely related to that for the fractional quantum Hall effect (FQHE) of the filling factor $\nu_c = 1/[p_1 - 1/(p_2 - \cdots - 1/p_N)\cdots]$. Under the chiral constraint the model describes the essential properties of the edge states for the FQHE with the above filling fraction. Furthermore the matrix deduced from the excitation spectrum characterizes the topological order of the FQHE state.

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It has been known that one-dimensional (1D) integrable quantum models with $1/r^2$ interaction may have a fundamental relationship [1,2] to the edge states of the fractional quantum Hall effect (FQHE) [3–5]. The construction of the eigenstates for the above models is indeed quite analogous to that for the FQHE; the ground state is given by the Jastrow-Slater wave function and the excited states are constructed by multiplying polynomials to the ground-state wave function [6–9]. Besides its own interest in the viewpoint of the integrability, therefore, this class of the quantum models is considered to exhibit interesting phenomena related to the FQHE [1,2].

In this paper we propose a novel hierarchy of the 1D SU(N) electron models with $1/r^2$ interaction, which has the close relationship to the hierarchical FQHE with the filling factor $\nu_c = 1/[p_1 - 1/(p_2 - \cdots - 1/p_N)\cdots]$. The spectrum, the bulk quantities, and the correlation exponents are computed by means of the asymptotic Bethe-ansatz (ABA) method [6,10]. We then clarify the relationship to the edge states of the FQHE in the disk geometry by imposing the chiral constraint [11,12]. In the classification of the energy spectrum we encounter the $N \times N$ matrix, and find that it coincides exactly with the matrix which specifies the topological order of the FQHE state with the above filling fraction ν_c [13].

We introduce a hierarchy of the Hamiltonians for the SU(N) electrons mutually interacting with $1/r^2$ potential in the 1D periodic ring of the circumference L,

$$\mathcal{H} = -\tfrac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha\le\beta}\sum_{i<j}\left(\frac{\pi}{L}\right)^2 \frac{\xi_\beta(\xi_\beta + P_{ij}^{\alpha\beta})}{\sin^2[\pi(x_i - x_j)/L]}, \tag{1}$$

where $\xi_\beta = \sum_{\gamma=1}^{\beta}\lambda_\gamma$ with $\lambda_\gamma \ge 0$, and $P_{ij}^{\alpha\beta}$ is the spin exchange operator with the spin indices $\alpha, \beta = 1, 2, \ldots, N$. We define the ith stage of the hierarchy ($i = 0, 1, \ldots, N$) by the set of parameters $\lambda_j > 0$ (=0) for $j \le i$ ($j > i$). Obviously the zeroth stage of the model describes the noninteracting SU(N) electrons. The first family is introduced by turning on the interaction λ_1 to the noninteracting Hamiltonian. Since the interaction λ_1 acts on every species of electrons, it only modifies the charge sector of physical quantities. The resulting model coincides with the SU(N) solvable Sutherland model [14,15]. Similarly the ith stage is defined by turning on the interaction λ_i to the $(i-1)$th stage of the Hamiltonian. The interaction λ_i acts on the particles with spins $\alpha = i, i+1, \ldots, N$, so that it generally affects the spin sector of the model. We will see that the construction of the present Hamiltonians is related to that for the FQHE [4,5,16]. Particularly our classification of the hierarchy is quite analogous to Jain's construction of the FQHE state out of the noninteracting model (zeroth order corresponds to the $\nu = N$ integer QHE) [5].

We start with the two-body scattering by the potential $D_{ij}^{-2}\lambda_1(\lambda_1+1)$ with $D_{ij} = (L/\pi)\sin[\pi(x_i - x_j)/L]$. In the asymptotic region ($|x_i - x_j| \gg 1$) this interaction gives rise to the phase shift function $\phi^{(1)} = \lambda_1 \operatorname{sgn}(k_i - k_j)$ for two momentums k_i and k_j. The stepwise form of $\phi^{(1)}(k)$ is inherent in the $1/r^2$ systems [6]. Taking into account the spin degrees of freedom in the interaction $D_{ij}^{-2}\lambda_1(\lambda_1 + P_{ij}^{\alpha\beta})$, the phase shift function is modified into $\phi^{(1)} + \phi_{\alpha\beta}^{(0)}$ where $\phi_{\alpha\beta}^{(0)}$ comes from the noninteracting SU(N) electrons; $\exp(-i\phi_{\alpha\beta}^{(0)}) = \lim_{\epsilon\to 0}(k_i - k_j - i\epsilon P_{ij}^{\alpha\beta})/(k_i - k_j - i\epsilon)$. The above phase shift function completely determines the two-body S matrix for the first family of the hierarchy, namely, for the integrable SU(N) Sutherland model. Consider further the scattering by the interaction (1) between particles with arbitrary spins $\alpha \le \beta$. We then find the phase shift function to be $\phi_{\alpha\beta}^{(1)}(k) = \xi_\beta \pi \operatorname{sgn}(k)$. Consequently the two-body S matrix for (1) is written down as

$$S_{ij} = e^{-i\phi_{\alpha\beta}^{(1)}(k_i - k_j)} e^{-i\phi_{\alpha\beta}^{(0)}(k_i - k_j)}, \tag{2}$$

where $\phi_{\alpha\beta}^{(1)}(k) = \xi_\beta \pi \operatorname{sgn}(k)$ for $\alpha \le \beta$. Imposing the periodic boundary conditions, consider now the many-body scattering problem. In the ABA approach the many-

body S matrix is conjectured to be decomposed into two-body matrices like the ordinary BA [6,10]. This conjecture works for all the integrable $1/r^2$ systems known so far [6–10]. In the present model, the situation is a bit more complicated since the phase shift $\phi^{(1)}_{\alpha\beta}(k)$ depends on the internal degrees of freedom. Fortunately it turns out that the treatment of $\phi^{(1)}_{\alpha\beta}(k)$ can be consistently embedded into the diagonalization procedure in the nested Bethe ansatz ($\phi^{(1)}_{\alpha\beta}$ takes the constant value in each step of nested BA). One consequently gets the ABA equations for the N kinds of the rapidities,

$$k^{(1)}_j L = 2\pi I^{(1)}_j + \sum_m \theta(k^{(2)}_m - k^{(1)}_j) + \lambda_1 \sum_l \theta(k^{(1)}_j - k^{(1)}_l), \tag{3}$$

$$(\lambda_\alpha + 1)\sum_l \theta(k^{(\alpha)}_m - k^{(\alpha)}_l) + 2\pi I^{(\alpha)}_m = \sum_{q=\pm 1}\sum_j \theta(k^{(\alpha)}_m - k^{(\alpha+q)}_j), \tag{4}$$

for $2 \le \alpha \le N$, where $\theta(k) = \pi\,\mathrm{sgn}(k)$ and $I^{(\alpha)}_j$ is an integer (or half-odd integer) which classifies the charge ($\alpha = 1$) and spin ($\alpha = 2, 3, \ldots, N$) excitations. The additional phase shift $\lambda_\alpha \pi\,\mathrm{sgn}(k)$ results from the first factor in (2). Here the number of electrons with spin α is denoted as M_α for $\alpha = 1, 2, \ldots, N$ ($M_{N+1} = 0$), and the number of the rapidities $k^{(\alpha)}_j$ is given by $\sum_{\tau=\alpha}^N M_\tau$. The total energy is expressed simply as $E = (1/2)\sum_j (k^{(1)}_j)^2$.

Let us consider the bulk properties in the thermodynamic limit. We introduce the density function $\rho_\alpha(k)$ for the rapidity $k^{(\alpha)}_j$. Converting Eqs. (3) and (4) into the integral equations, one finds that the integral kernel is given by the δ function, which results in the remarkable fact that the density function has the constant value in the region of $[R_\tau : Q_{\tau+1} < |k| < Q_\tau]$ where $Q_1 \ge Q_2 \ge \cdots \ge Q_N$ with $Q_{N+1} = 0$, while $\rho_\alpha(k) = 0$ otherwise. The density function $\rho^{(\tau)}_\alpha(k)$ in the region R_τ thus takes the value

$$\rho^{(\tau)}_\alpha(k) = \frac{1}{2\pi}\prod_{i=1}^{\alpha}{}', \frac{1}{p_i - \dfrac{1}{p_{i+1}\cdots - \dfrac{1}{p_\tau}}}, \quad \tau \ge \alpha, \tag{5}$$

where we have used the new parameters $p_i = \lambda_i + 2 - \delta_{i1}$. Introducing the $(\tau-\alpha)\times(\tau-\alpha)$ matrix $\mathbf{T}^\tau_\alpha$ defined by

$$(T^\tau_\alpha)_{ij} = p_i\delta_{ij} - \delta_{i(i\pm1)}, \tag{6}$$

for $i, j = \alpha, \alpha+1, \ldots, \tau$, the formula (5) is cast into a succinct form, $\rho^{(\tau)}_\alpha(k) = (1/2\pi)\det\mathbf{T}^\tau_{\alpha+1}/\det\mathbf{T}^\tau_1$. The matrix $\mathbf{T}^\tau_\alpha$ plays a central role in the following analysis. Here we introduce the important key quantity $\nu_c = 2\pi\rho^{(N)}_1$ which characterizes the hierarchical stage of the model,

$$\nu_c = \frac{1}{p_1 - \dfrac{1}{p_2\cdots - \dfrac{1}{p_N}}}. \tag{7}$$

Note that the spacing of the charge rapidities $k^{(1)}_j$ is enlarged $1/\nu_c$ times as large as the free fermion case due to the repulsive interaction. This may imply that the effective volume of the electron is considered to be $1/\nu_c$ times as large as that for the free fermion. It will be shown that ν_c coincides with the filling factor of the FQHE if we apply the model to the edge states by imposing the chiral constraint.

We summarize the results for the static quantities here. The ground state is given by setting $Q_1 = Q_2 = \cdots = Q_N$. The corresponding energy is computed as $E_g/L = \pi^2 n^3/6\nu_c^2$, where $n = \sum_{j=1}^N n_j$ is the total electron density. Note that for the ith stage of the hierarchy, each species of electrons has different densities in general: $n_1 > \cdots > n_{i-1} > n_i = n_{i+1} \cdots = n_N$. From the second derivative of $E_g(n)$, we then get the compressibility in terms of ν_c,

$$\kappa_c = (\nu_c/\pi)^2 n^{-1}, \tag{8}$$

which depends linearly on n^{-1} as in the noninteracting case. Following the method developed by Yang and Yang [17], the free energy F at finite temperatures (T) is determined by the formula $F = -\mu - (T/2\pi)\int \ln\{1 + \exp[-\epsilon_1(k)]\}dk$ in terms of the dressed energy function

$$\epsilon_1(k)/T = \tfrac{1}{2}k^2 - \mu - \ln\{1 + \exp[-\epsilon_2(k)]\} + \lambda_1 \ln\{1 + \exp[-\epsilon_1(k)/T]\}, \tag{9}$$

$$\epsilon_\alpha(k)/T = \sum_{q=-1,0,1} (-1)^q(1 + \lambda_\alpha\delta_{q0}) \times \ln\{1 + \exp[-\epsilon_{\alpha+q}(k)]/T\} \tag{10}$$

for $\alpha = 2, 3, \ldots, N$, with $\epsilon_{N+1} = \infty$. The standard Sommerfeld expansion gives the heat capacity at low temperatures, $C/T = (\pi/3)\sum_{\alpha=1}^N (1/v_\alpha)$ with the velocity

$$v_\alpha = \pi n/2\nu_c, \tag{11}$$

where v_1 corresponds to the charge velocity and others to the $N-1$ kinds of spin velocities. We note that all the velocities take the same value irrespective of the interaction strength. The Luttinger liquid relation between the compressibility and the charge velocity reads $\pi\kappa_c v_1 = \nu_c/2$, which essentially controls the critical behavior of the charge excitation [18]. It is now easy to obtain the excitation spectrum using the fact that the velocities (11) are all the same and the phase shift function takes the simple form of $\mathrm{sgn}(k)$. The excitation energy is thus classified in terms of the velocity v,

$$\Delta E = \frac{2\pi v}{L}\left(\frac{1}{4}\vec{m}^t\mathbf{T}^N_1\vec{m} + \vec{d}^t(\mathbf{T}^N_1)^{-1}\vec{d}\right), \tag{12}$$

where the matrix $\mathbf{T}_1^N$ is defined in (6). Note that (12) takes the modular invariant form consistent with the conformal invariance [19]. Here $\vec{m}$ is the N-component vector out of quantum numbers which specify the charge (m_1) and spin (m_α, $\alpha \geq 2$) excitations. The vector $\vec{d}$ consists of quantum numbers which carry the current $2d_\alpha k_F$ with the Fermi momentum k_F. Following the finite-size scaling in conformal field theory [20], we easily get the critical exponent η for correlation functions by the formula $\eta = \Delta E/(\pi v/L)$ by choosing the appropriate quantum numbers (for details see [10,21]).

We now wish to observe what happens if these interacting electrons move through the 1D lattice. We introduce a hierarchy of the lattice models [SU(N) t-J model] with $1/r^2$ interaction,

$$\mathcal{H} = \sum_{\alpha,i\neq j} D_{ij}^{-2} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{\alpha\leq\beta, i<j} D_{ij}^{-2}\xi_\beta(\xi_\beta + P_{ij}^{\alpha\beta}), \quad (13)$$

where $D_{ij} = (L/\pi)\sin[\pi(x_i - x_j)/L]$, $\xi_\alpha = \sum_{\gamma=1}^{\alpha} l_\gamma$, $l_\alpha = (p_\alpha - 2 + \delta_{\alpha 1})$, and the configurations with more than one electron at every site are forbidden. The Hamiltonian (13) is a hierarchical extension of the model discussed in [14]. We note that the kinetic energy with the hopping D_{ij}^{-2} is simply given by $E = \text{const} + (1/2)\sum_j k_j^2$ in the noninteracting case. The two-body S matrix for this model is found to take the same form as (2), and thus the ABA equations are given in the formulas (3) and (4) replacing λ_α by l_α. As a result the bulk quantities and the excitation spectrum are obtained in the same expressions (8), (11), and (12) [22]. In this case, however, there are two important constraints due to the lattice effects: (a) a parameter l_α is to be an even integer [23], and (b) the available range of the rapidity is restricted to $-\pi \leq k_j^{(\alpha)} \leq \pi$ [24]. From (a) the parameter ν_c defined in (7) becomes the fraction with the odd denominator, which reveals the analogy to the FQHE more explicitly. Moreover the constraint (b) provides a rather striking result. As mentioned in (7) the spacing of the charge rapidities is uniformly enlarged ($1/\nu_c$ times) due to the repulsive $1/r^2$ interaction. This fact together with the constraint (b) gives rise to a singular behavior of the system at the electron density $n = \nu_c$ where the normal fluidlike state breaks down for $n > \nu_c$ [25]. This phenomenon seems to imply that the singularity for the noninteracting SU(N) lattice electrons at the density $n = N$, where the semiconductor gap opens, is modified into the singularity at the filling $n = \nu_c$ in the presence of the $1/r^2$ interaction. If we turn off the interaction, the critical density indeed becomes $\nu_c = N$. It is instructive to note that the above interpretation is quite similar to that for the evolution of the FQHE state starting from the integer QHE state with the filling N [5].

In order to observe the relationship to the FQHE more explicitly we now discuss the effects of the chiral constraint for the metallic phase of the lattice model (13), and confront the results with the chiral Luttinger liquid for the FQHE in the disk geometry [12]. We first note that the stepwise form of the phase shift function implies that there is no marginal operator in the theory [7]. For example, there is no logarithmic correction in the correlation functions. As a consequence of the stepwise phase shift, therefore, the Umklapp scattering is irrelevant in the present model [7,26], and we can separate right- and left-going waves without changing the conformal dimensions. By suppressing the excitations which carry the current in (12), one thus gets the excitation energy for the right- (or left-) going sector as

$$\Delta E = \frac{\pi v}{L}\vec{m}^t \mathbf{T}_1^N \vec{m}. \quad (14)$$

Using the finite-size scaling [20], we can read all the critical exponents for the correlation functions from (14). The obtained results reproduce those for the chiral Luttinger liquid for the edge states of FQHE with the filling ν_c [12]. For example, by choosing $m_\beta = 1$ $(= 0)$ for $\beta \leq \alpha$ $(\beta > \alpha)$, the critical exponent θ_α for the momentum distribution function is given as $\theta_\alpha = \sum_{j=1}^{\alpha} l_j = \sum_{j=1}^{\alpha}(p_j - 2 + \delta_{j1})$. We emphasize here that the matrix $\mathbf{T}_1^N$ is the key quantity which characterizes the critical behavior of the present model both for the chiral and nonchiral cases. In particular the effective field theory deduced from the spectrum (14) is essentially the same as that for the chiral Luttinger liquid [12] in which the matrix $\mathbf{T}_1^N$ specifies the topological order of the bulk FQHE with the filling fraction ν_c [27]. For example, the first stage of the hierarchy with $l_1 > 0$ and $l_\alpha = 0$ for $\alpha \geq 2$, the matrix $\mathbf{T}_1^N$ characterizes the topological order for the filling fraction $\nu_c = N/(Nl_1 + 1)$ [5,13,28]. Similarly the general case of the spectrum (14) produces the effective field theory for the edge states of the hierarchical FQHE discussed in [13]. We mention here that the formula for the nonchiral case (12) may also be relevant to the edge states of the FQHE in the case of the cylinder geometry in which the right- and left-going waves in the opposite edges are mixed through certain scattering processes [12].

We have seen so far that the above hierarchical models involve the fundamental properties expected for the edge states of the FQHE with the filling fraction ν_c. We stress that this correspondence is not accidental, the origin of which is clearly seen by observing the role played by the phase shift function $l_\alpha \pi\,\text{sgn}(k)$. According to the composite fermion approach to the FQHE [5] or the corresponding effective field theory [13,28], one may attach l_α flux quanta to the electron in order to evolve the hierarchy of the FQHE state. An important point is that attaching the l_α flux quanta to particles in the FQHE corresponds to introducing the phase shift function $l_\alpha \pi\,\text{sgn}(k)$ in the present 1D systems, which can be plausibly reproduced by the $1/r^2$ interaction. This correspondence may clarify why the family of the models (1) and (13) reproduces the evolution of a certain hierarchy of the FQHE state so remarkably.

We conclude this paper by briefly mentioning a possible way to construct eigenfunctions which reproduce

the ABA spectrum. Following the construction of the eigenstates for the known $1/r^2$ models, one naturally expects that the Jastrow-product wave function gives the ground state, and the polynomials of Calmeyer-Laughlin type describe excited states. In fact we have checked that the Jastrow-product wave function out of the parameters $\xi_{\alpha\beta}$, $\psi = \prod_{l,m} |e^{2\pi i x_l} - e^{2\pi i x_m}|^{\xi_\beta - \delta_{\alpha\beta}} \psi_G$ provides the exact same critical exponents deduced from (12) and (14), where ψ_G is the SU(N) completely projected Gutzwiller wave function. So, the wave functions with polynomials multiplied to ψ are expected to give the set of eigenfunctions for (1). The construction of the eigenfunctions, which may ensure the ABA-spectrum microscopically, will be reported elsewhere.

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