Statics of a "Self-Organized" Percolation Model

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A stochastic "forest-fire" model is considered. Sites are filled individually at a constant mean rate; also, "sparks" are dropped at a small rate k, and instantaneously burn up the entire cluster they hit. I find *nontrivial* critical exponents in the self-organized critical limit $k \rightarrow 0$, contrary to earlier results of Drossel and Schwabl. Spatial correlation functions and a site occupancy correlation exponent are measured for the first time. Scaling relations, derived by analogy to uncorrelated percolation, are used extensively as numerical checks. Hyperscaling is violated in this system.

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"Self-organized criticality" (SOC) [1] is a paradigm for the way in which some strongly nonequilibrium systems develop self-similar (power-law) correlations in the steady state, without the need to tune a parameter to a special critical value. It might ultimately explain the ubiquity of fractal patterns in nature [2]. The best studied model systems are various "sandpile" models on lattices.

A conservation law (as in "sandpiles") is commonly considered essential for SOC. One school, analyzing field theory versions of the models with stochastic driving terms, concludes that power laws emerge in SOC much as they do in the (trivial) case of noninteracting diffusing particles; however, the exponents are determined by the imposed current (which selects a direction) plus the nonlinearities [3,4]. A second school finds that a parameter *does* need to be tuned to a critical value at which the diffusion constant diverges; however, this value is generically adopted throughout the system to satisfy the boundary conditions with a finite current [5]. The model presented below *lacks* conservation laws, but exhibits nontrivial criticality as the control parameter approaches *zero.* This is SOC in a weak sense, since small values of the parameter could occur generically in nature.

Here I study a stochastic model evolving in continuous time [6]. Each lattice site has two possible states, "filled" or "empty," and there are two dynamical rules: (i) Sites, chosen at random, are filled with average rate unity per site (if the site is already full, nothing happens); (ii) at an average rate per site $k \ll 1$, "sparks" are dropped at random; if a spark falls on a filled site, the entire cluster of sites connected to it "burns" and is reset to empty. The dynamics depends directly on connectivity, and the exponents have the same definitions as in percolation models (but different values), so I call this a "self-organized percolation" model. One possible system for realizing a similar model scenario would be a surface chemical reaction with a very local autocatalysis [7] (involving neither heat diffusion, atom diffusion, nor reconstructions). Similar dynamic percolation models also arise in the modeling of "patchiness" in ecology [8] (due to, e.g., epidemics or real forest fires).

In the earlier "forest-fire" model [9,10], sites had a third possible state, "burning." Burning sites rapidly set their neighbors burning at an average "spreading rate" $\sim w$, and then become empty. The self-organized percolation model is just the limiting case $w \rightarrow \infty$. Much previous SOC work focused on the critical *dynamics*: If $w < \infty$, the duration of an avalanche or fire scales with its size, so spatial power laws imply time power laws [1,9]. Here, however, we have eliminated such long time scales since every fire is instantaneous. The longest other time scale is the spacing between successive fillings of a site, which is of order unity. Therefore, my focus here is on the *statics.*

If we did let $w < \infty$, then a fire could sustain itself indefinitely without sparks ($k=0$) [9,10]. This case was found to be uninteresting [10]: The steady state is just a succession of "fire" fronts, which are ordinary curves (with trivial fractal dimension $D=1$). On the other hand, Drossel and Schwabl recently showed [11] that the self-organized percolation model exhibits SOC (in the weak sense) as $k \rightarrow 0$. They argued, analytically and numerically, that the exponents have simple integer values. However, the analytical argument of Ref. [11] is not valid, and my numerical results reported below—using k roughly an order of magnitude closer to criticality than theirs—show different exponent values. This indicates a new universality class which is quite different from ordinary percolation.

Simulations and cluster distribution.—Simulations were performed on a square lattice with $N=L^2$ sites for $L=128,256,512$ (results shown here are all at $L=512$) with $k=1/2^n$ down to 1/2048. A typical run was $\sim 10^3$–10^4 passes, where a pass is one attempted fill step per site.

Let $m_i=1$ for an occupied site and 0 otherwise; thus the occupied fraction is $p \equiv \langle m_i \rangle$ which has a maximum value $p_c \equiv \lim_{k\rightarrow 0} p(k)$. Define m_i^* to be the size of the cluster to which site i belongs. Note that $\sum_i m_i^* = \sum_l S_l^2$, where $\{S_l\}$ are the sizes (number of sites) of all the disjoint clusters. Hence

$$\chi \equiv N^{-1}\left\langle \sum_l S_l^2 \right\rangle = \langle m_i^* \rangle . \quad (1)$$

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Now, given an instantaneous configuration $\{m_i\}$, it is easily seen that $\langle dm_i/dt \rangle = -km_i^* + (1-m_i)$, where the terms represent burning and filling, respectively. In the steady state dm_i/dt must average to zero [11], so

$$k\chi = (1-p) \tag{2}$$

[equivalent to Eq. (1) of Ref. [11]]. The parameter k is the same as "f/p" in Ref. [11]; also, $\chi = \langle \sigma_i \rangle$ is the same as "$\bar{s}$" in Ref. [11]. Since $p_c < 1$ in $d > 1$, (2) implies that $\chi \to \infty$ as $k \to 0$. By (1), that is possible only if arbitrarily large clusters appear, reflecting some sort of criticality.

Define $\tilde{\gamma}$ via $\chi \sim k^{-\tilde{\gamma}}$. (I write exponents of the control parameter k with a tilde to distinguish them from exponents of $p_c - p$.) Then (2) implies

$$\tilde{\gamma} \equiv 1 \, . \tag{3}$$

This trivial exponent relation is the source of most of the nontrivial ones derived below. Numerically $\tilde{\gamma} = 1.03 \pm 0.03$ is found, providing a small check of the accuracy of the results and the estimation of error bars. We also define γ by $p_c - p \sim k^{1/\gamma}$ equivalent, via (2), to the usual $\chi \sim (p_c - p)^{-\gamma}$. Numerically $p_c = 0.411 \pm 0.002$ (compare $p_c \approx 0.39$ in Ref. [11]).

Consider $n(s)$, the number density (per site) of clusters of size s. The probability that a spark burns up s sites is $n^*(s) \equiv sn(s)$. This is one of several quantities written with stars and called "burn weighted," because they are measured during burning and hence weighted by cluster size (i.e., by m_i^* or s). Let us make the usual scaling hypothesis

$$n(s) \sim s^{-\tau} f_n(s/S_{\max}) \tag{4a}$$

with the approximate cutoff at $S_{\max} \sim k^{-X}$, for some X. What I actually plotted (Fig. 1) and analyzed is $n_>^*(S) \equiv \sum_{s>S} n^*(s)$, the *cumulative* burn-weighted distribution. From (4), its scaling form is

$$n_>^*(S) \sim S^{-(\tau-2)} F_{n*}(Sk^X) \, . \tag{4b}$$

Figure 1 also shows the average radius of gyration $R_g(s)$ as a function of cluster size s [averaged over all clusters that fell in the bin $(s, \sqrt{2}s)$]. I assume that the

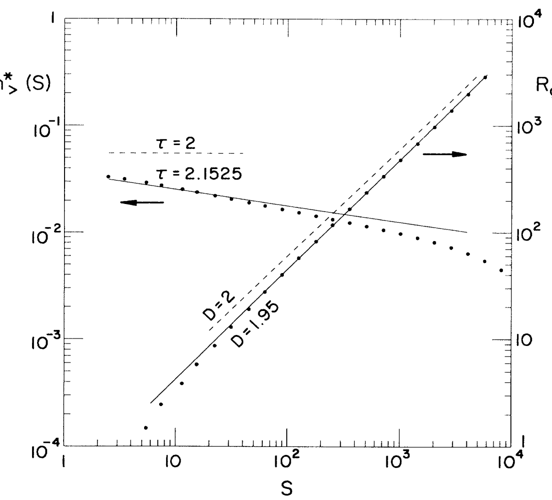

FIG. 1. Cumulative burn-weighted cluster frequency $\eta_>^*(S)$ and mean squared radius of gyration $R_g^2(S)$ as a function of cluster size S. The solid curves are for spark rate $k = 2^{-11} \approx 0.48 \times 10^{-3}$.

clusters have fractal dimension D in the scaling regime, i.e., $R_g \sim s^{1/D}$, implying a scaling form

$$R_g^2 \sim s^{2/D} f_R(sk^X) \, . \tag{5}$$

The values of τ and X from scaling plots of (4b), and of D from scaling plots of (5), are in Table I. Clearly $\tau \neq 2$, contrary to Ref. [11]. The reasoning which led to $D = \tau = 2$ in Ref. [11] [see their Eq. (5)] is clearly incorrect: Ordinary percolation satisfies all their assumptions, but has nontrivial D, τ. The misstep in Ref. [11] is the assumption that $\langle s \rangle \equiv \chi \propto S_{\max}$; really, $\chi \propto S_{\max}^{1/X}$.

Correlation functions.—I have defined three different correlation functions. Given a configuration $\{m_i\}$, let $K(i,j) = 1$ (0) if sites (i,j) are connected (unconnected). Then, as in ordinary percolation, the connectivity correlation function is $K(\mathbf{r}_{ij}) \equiv \langle K(i,j) \rangle$. Furthermore, the burn-weighted connectivity correlation function is $K^*(\mathbf{r}_{ij}) \equiv \langle m_i^* K(i,j) \rangle$. Finally, long-range occupancy correlations develop, because burning events simultaneously remove many nearby sites. So I define the site oc-

TABLE I. Critical exponents.

Exponent	Measured	Proposed	Alternate	Uncorrelated	Ref. [11]
$\tilde{\nu}$	0.58(0.02)	0.605	$3/5 = 0.60$	$24/43 = 0.558$	"ν"$\equiv 1/d = 0.50$
$\eta = 2 - 1/\tilde{\nu}$	0.411(0.02)	0.347	$1/3 = 0.333$	$5/24 = 0.208$	0
D	1.95(0.025)	1.95	2	$91/48 = 1.896$	"μ"$\equiv d = 2$
$\eta_* = 2 - D$	0.05(0.01)	0.05	0	$5/48 = 0.104$	0
$\tau - 2 = 1 - 1/X$	0.150(0.005)	0.1525	$1/6 = 0.167$	$5/91 = 0.055$	0
$X = D\tilde{\nu}$	1.167(0.015)	1.180	$6/5 = 1.2$	$91/86 = 1.058$	"λ" = 1
$1/\gamma$	0.41(0.01)	0.41[a]	0.41[a]	$18/43 = 0.419$	?
$\eta_{\rm occ}$	0.120(0.015)	0.120[a]	0	"∞"	?

[a]Set equal to measured value.

cupancy correlation function $K_{occ}(\mathbf{r}_{ij}) \equiv \langle m_i m_j \rangle - p^2$.

We expect the correlation functions to exhibit scaling forms

$$K(r) \sim r^{-(d-2+\eta)} f(r/\xi) , \tag{6}$$

$$K^*(r)/\chi \sim r^{-(d-2+\eta^*)} f^*(r/\xi^*) , \tag{7}$$

$$K_{occ}(r) \sim r^{-(d-2+\eta_{occ})} f_{occ}(r/\xi_{occ}) . \tag{8}$$

Note that, as $k \to 0$, $K^*(\mathbf{r})$ is divergent; however, its ratio with $\chi \equiv K^*(0)$, as in (10), approaches a limit.

We expect critical behavior $\xi \sim k^{-\tilde{\nu}}$ (similarly for ξ^* and ξ_{occ}). Furthermore, we expect

$$\tilde{\nu} \equiv \tilde{\nu}^* \equiv \tilde{\nu}_{corr} . \tag{9}$$

The measured correlation functions are shown in Fig. 2. From scaling plots, $\tilde{\nu} = 0.60$, $\tilde{\nu}^* = 0.58$, and $\tilde{\nu}_{occ} = 0.56$. These results are consistent with (9) and so I have put a single value in Table I.

In fact, Fig. 2 includes correlation functions in both the $\langle 10 \rangle$ and $\langle 11 \rangle$ directions, but these are indistinguishable. Evidently, the correlations develop rotational symmetry at criticality. The reason for this seems mysterious; it cannot arise from a diffusion propagator (no conservation law), and the model cannot be transcribed into a field theory (the cluster burn step implies infinite-range interactions).

Scaling relations and results.— As in equilibrium systems, (6) implies a susceptibility sum rule:

$$\chi = \sum_{\mathbf{r}} K(\mathbf{r}) \sim \xi^{2-\eta} , \tag{10}$$

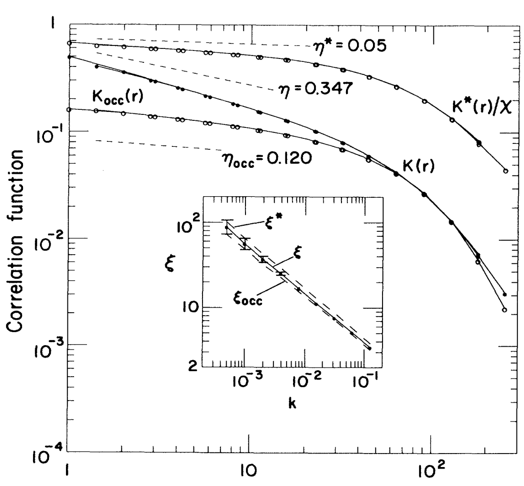

FIG. 2. Correlation functions for connectedness $K(r)$, burn-weighted connectedness $K^*(r)/\chi$, and occupancy $K_{occ}(r)$, for $k = 2^{-11} \approx 0.48 \times 10^{-3}$. The dashed lines have the slopes as labeled, characterizing the initial power-law decay. The inset shows the three correlation lengths as a function of spark rate k. Where no error bars are shown for $\xi(k)$, they are smaller than the dots; ξ^* and ξ_{occ} have comparable error bars.

i.e., $\tilde{\gamma} = (2-\eta)\tilde{\nu}$, or [via (3)]

$$2 - \eta = 1/\tilde{\nu} . \tag{11}$$

On the other hand, using (4) gives $\chi = \sum_s s^2 n(s) \sim k^{-(3-\tau)X}$ which implies [via (3) again] the exponent relation

$$X = (3-\tau)^{-1} . \tag{12}$$

Again, as in (7) we could write a burn-weighted susceptibility $\chi^* \equiv \sum_{\mathbf{r}} K^*(\mathbf{r})/X \sim \xi^{2-\eta^*}$ using (7); on the other hand, $\sum_{\mathbf{r}} K^*(\mathbf{r}) = \langle \sum_l S_l^3 \rangle / N$ so

$$\chi^* = \sum_s s^3 n(s) \Big/ \sum_s s^2 n(s) \sim S_{max} .$$

The cutoff radius of the scaling cluster distribution is $\sim \xi$, so $S_{max} \sim \xi^D$; but also $S_{max} \sim k^{-X}$ [as in Eqs. (4) and (5)]. Hence

$$D = 2 - \eta^* , \tag{13}$$

$$D\tilde{\nu} = X . \tag{14}$$

All in all, there are four independent exponents which I will take to be D, $\tilde{\nu}$, γ, and η_{occ}. The first six lines of Table I give independent measurements of D, $\tilde{\nu}$, and the four additional exponents related to them by the scaling relations [(11)–(14)] [12]. Harmonizing these results (forcing the scaling relations to hold) gave $\tilde{\nu} = 0.605 \pm 0.01$ and $D = 1.95 \pm 0.01$, and the other values in column 3. Note that the most precise values of $\tilde{\nu}$ and D are provided, via η and η^*, by measuring of the correlation functions. (Perhaps this would be useful in other systems for which scaling relations are known.) The last two exponents are measured only one way so the scaling relations do not provide any checks on them.

The possibility of $D = 2$ cannot be completely ruled out, so I also made an alternate harmonization (column 4) with $D \equiv 2$ fixed at the start; the resulting value of $\tilde{\nu}$ was so close to 3/5 that I chose to present this rational value. Independently, I chose to record a trivial value of η_{occ} since that too cannot be ruled out. The last two columns give the exact exponent values for ordinary (uncorrelated) percolation, and the values proposed for (essentially) the present model by Drossel and Schwabl [11] (their notations given in quotes). The deviations of the harmonized values from the measured ones are typically 1.5 times the estimated error; for the alternate harmonization with $D \equiv 2$, they are twice as large. By comparison, the uncorrelated-percolation exponents give a bad fit; an even worse fit is given by the proposed exponents of Ref. [11] (except D).

The exponent $1/\gamma$ is indistinguishable from that of uncorrelated percolation. However, $\tilde{\nu}$ and D are both moderately larger, so their product (and hence τ) is much larger than in the uncorrelated case. Of course, η_{occ} is totally different. (We must call it "∞" in the uncorrelated case, since correlations die off infinitely quickly.)

The *hyperscaling* relation is

$$\theta \equiv 2D - d - 1/\tilde{\nu} \equiv 2 - d + \eta - 2\eta^* = 0 . \quad (15)$$

It is satisfied for ordinary percolation [use $D \equiv d - \beta/\nu$ and $1/\tilde{\nu} \equiv \gamma/\nu$ to put (15) in a more familiar hyperscaling form]. But (15) is *violated* by our self-organized percolation model (see Table I). To interpret (15), we must review a property of uncorrelated percolation. Cut a box of diameter $l \gg 1$ out of the system. A cluster big enough to span the box has $S_l \sim l^D$ sites; the number density of such clusters is $n_>(S_l) \equiv \sum_{s>S_l} n(s) \sim (S_l)^{-(\tau-1)}$ using (4a). Thus, the total number of clusters in a box which are large enough to span it scales as

$$l^d n_> S_l \sim l^{d-(\tau-1)D} \equiv l^{-\theta} \quad (16)$$

using (12), (14), and (15) for the last equality. Thus, the validity of (15) would indicate the spatial distribution of clusters is homogeneous: At criticality, every box has a spanning cluster (or a piece of a larger cluster). On the other hand, the violation of (15) suggests a cluster distribution like that of the "fractal curdling" process [2]: Most boxes contain no large pieces; however, when they do the fractal geometry is always the same.

I also investigated the one-dimensional self-organized percolation model, which is not trivially soluble. The correlation function $K(r)$ does not decay exponentially, but seems to have logarithmic corrections, somewhat reminiscent of the "multifractal" scaling [13,14].

Socolar, Grinstein, and Jayaprakash [15] studied a toy forest-fire model, in which my variable m_i is replaced by a continuous "height" $u_i(t)$, with a *deterministic* dynamics. The height grows at a steady rate p_{grow} which is the analog of $1/w$ in forest-fire models; a tree becomes burnable at $u_i = 1$ (analog of my empty $\rightarrow$ filled step); and it catches fire spontaneously at $u_i = U$ (analog of my spark step). The parameter $1/U$ is thus analogous to my k, and I conjecture that SOC should occur in the limit $p_{\text{grow}} \rightarrow 0$, $U \rightarrow \infty$. The simulations of Ref. [15], which saw no evidence of SOC, took only $U = 2$.

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Note added.—Grassberger [16] has obtained (in my notation) $p_c = 0.4075$, $\tilde{\nu} = 0.584(0.01)$, $D = 2$, $\tau - 2 = 0.15(0.02)$, $X = 1.08(0.02)$, and $1/\gamma = 1.2$ [these do not agree with the scaling laws (12) and (14)]. Reference [16] is a much larger simulation than mine (~ 10 times as many steps/site, and 256 as many sites); however, I believe my quality is comparable, because unlike Ref. [16] I used correlation functions, which seem to give more precise exponents. The observation in Ref. [16], p. 2087, that "locally the state is nearly everywhere away from the instability threshold," seems related to my remarks about the geometrical viewpoint on hyperscaling and fractal curdling.

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