Statistical Balance of Vorticity and a New Scale for Vortical Structures in Turbulence

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The balance of one-point and two-point statistical characteristics of vorticity is considered on the basis of the Navier-Stokes equations. It is shown that within the inertial range of scales ($L\,\mathrm{Re}^{-3/4} \ll r \ll L$, L external scale, Re Reynolds number) there is a physically distinguished scale $l_s \sim L\,\mathrm{Re}^{-3/10}$. The balance of vortical correlations with scales $r \gtrsim l_s$ is directly affected by the large-scale motion. l_s is a natural length scale for the "vortex strings," observed experimentally and numerically in three-dimensional turbulent flows. The twist of vortex lines in the internal structure of vortex strings is also briefly discussed.

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The dynamics of turbulent flows is better understood in terms of local characteristics of motion [1], which have a mechanism of self-amplification. For three-dimensional turbulence the local characteristic is the vorticity field and self-amplification is due to the effect of stretching of vortex filaments [2,3]. For two-dimensional turbulence the local characteristic is the vorticity gradient [4]. We use the concept of self-amplification, because in both cases the deformation tensor, responsible for amplification, is expressed in terms of local characteristics [3-7]. The result of conditional averaging of the deformation tensor with fixed vorticity indicates a definite statistical tendency to the formation of "vortex strings" in three-dimensional turbulence [6,7]. Laboratory observation of such elongated vortices was reported in Ref. [8] (see also references therein for numerical experiments).

It seems natural to assume that the characteristic length of vortex strings depends on the external scale L and Reynolds number Re of the flow. It was shown [7] that for high Re the effect of large-scale motion on the statistical balance of enstrophy is $\sim \mathrm{Re}^{-3/2}$ and can be neglected. We will show below that the same is true for high order one-point characteristics of vorticity. However, the balance of vortical correlations within the inertial range [Eq. (17)] is affected by the large-scale motion. This leads to a new characteristic scale, which we associate with vortex strings.

Consider equations for a three-dimensional vorticity field in an incompressible fluid, which follow from the Navier-Stokes equations:

$$\frac{\partial \omega_i}{\partial t} + v_k \frac{\partial \omega_i}{\partial x_k} = \frac{\partial v_i}{\partial x_k}\omega_k + \nu \Delta \omega_i + \phi_i, \quad \frac{\partial v_i}{\partial x_i} = 0, \quad (1)$$

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}, \quad \phi_i = \epsilon_{ijk}\frac{\partial f_k}{\partial x_j}, \quad \frac{\partial f_i}{\partial x_i} = 0. \quad (2)$$

Here, v_i, ω_i, and f_i are correspondingly velocity, vorticity, and external force, ϵ_{ijk} is the unit antisymmetric tensor, and ν is kinematic viscosity. Assume that turbulent flow is statistically stationary, homogeneous, and isotropic. The energy, supplied by large-scale random forces, dissipates due to viscosity with the mean rate,

$$\epsilon = \langle f_i v_i \rangle = \nu \langle \omega_i^2 \rangle . \quad (3)$$

Here $\langle\ \rangle$ means statistical averaging and all fields are taken at the same space-time location. Assume also that forces are Gaussian and δ correlated in time, which gives the formula [9,10]

$$\langle \phi_i(\mathbf{x}) R[\boldsymbol{\omega}(\cdot)] \rangle = \frac{1}{2}\int d^3x' \Phi_{ij}(\mathbf{x}'-\mathbf{x}) \left\langle \frac{\delta R[\boldsymbol{\omega}(\cdot)]}{\delta \omega_j(\mathbf{x}')} \right\rangle . \quad (4)$$

Here R is any functional of vorticity field, Φ_{ij} is the space correlation tensor of ϕ_i, δ corresponds to functional derivative, and all fields are taken at the same time.

By multiplying (1) with $\omega_i \omega^n$ $(n \geq 0)$, averaging, using (4), and simple manipulations, we get the statistical balance of one-point characteristics of vorticity:

$$\left\langle \frac{\partial v_i}{\partial x_k}\omega_k \omega_i \omega^n \right\rangle = \nu \left\langle \omega^n \left[\left(\frac{\partial \omega_i}{\partial x_k}\right)^2 + n\left(\frac{\partial \omega}{\partial x_k}\right)^2 \right] \right\rangle - \frac{3+n}{6}\Phi_{ii}(0)\langle \omega^n \rangle . \quad (5)$$

For $n=0$ it is just a balance of enstrophy [7]. The left-hand side of (5) represents the effect of stretching, the first term on the right-hand side (rhs) of (5) is viscous smoothing, and the last term corresponds to the influence of large-scale motion, supplying energy. Using (2), we have

$$\Phi_{ii}(r) = -\frac{\partial^2 F_{ii}(r)}{\partial r_k^2} = -F''(r) - \frac{2}{r}F'(r), \quad F \equiv F_{ii}, \quad (6)$$

where $F_{ij}(\mathbf{r})$ is the space correlation tensor of f_i and the prime indicates differentiation over r. Formula (4) with substitution f_i and v_i instead of ϕ_i and ω_i gives [9]

$$\langle f_i(\mathbf{x}) v_j(\mathbf{x}') \rangle = \tfrac{1}{2}F_{ij}(\mathbf{x}'-\mathbf{x}), \quad F_{ii}(0) = 2\epsilon . \quad (7)$$

Function $F(r)$ is even (isotropy), thus

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$$\Phi_{ii}(0)=6\epsilon L^{-2},\ \ L^{-2}=-F''(0)[F(0)]^{-1},\qquad\qquad (8)$$
$$\mathrm{Re}=\epsilon^{1/3}L^{4/3}\nu^{-1},$$

where L is the natural external scale for this turbulent flow [9] and Re is the corresponding Reynolds number. This result can be obtained by dimensional argument for a broad class of large-scale forces. Taking into account that the characteristic value of vorticity and the velocity gradient is $\sim\epsilon^{1/2}\nu^{-1/2}$ (3), we see that the relative contribution of large-scale motion in (5) is $\sim\mathrm{Re}^{-3/2}$ and can be neglected for $\mathrm{Re}\gg 1$. Thus the balance of one-point statistical characteristics of vorticity, including high order moments, is not sensitive to the large-scale motion, provided that the level of energy dissipation is supported (3) and $\mathrm{Re}\gg 1$. This conclusion is also important as additional justification of the equation for the conditionally averaged vorticity field $\bar{\Omega}_i(\mathbf{r},\boldsymbol{\omega})$ a distance $\mathbf{r}$ from a point with fixed vorticity $\boldsymbol{\omega}$ [5-7]. The left-hand side of (1), being conditionally averaged with fixed $\boldsymbol{\omega}$ at the same point, gives zero, which is proven by multiplying it with $A(\omega)\omega_i$ with arbitrary function $A(\omega)$ and averaging unconditionally [compare with [7], formulas (7)-(10), where proof is slightly different]. The first two terms on the rhs of (1) give the integral equation (or relation) for $\bar{\Omega}_i$. The last term (being conditionally averaged with fixed $\boldsymbol{\omega}$ at the same point), according to (4), (5), and (8), is acting as an operator,

$$\bar{\phi}_i=\epsilon L^{-2}\frac{\partial}{\partial\omega_i},\qquad (9)$$

and can be neglected for large Re. Let us note that the balance of one-point characteristics of vorticity (5) involves integrals of the two-point characteristic of vorticity $\bar{\Omega}_i(\mathbf{r},\boldsymbol{\omega})$. The general expression obtained in [5,7] for the field $\bar{\Omega}_i(\mathbf{r},\boldsymbol{\omega})$ reveals a twist of vortex lines, corresponding to the balance between stretching and viscous smoothing [see formula (23) and below in Ref. [7]]. This statistically important twist probably contributes to the helical shape of explosion of vortex strings, when they become unstable [8].

Now we turn to the balance of two-point characteristics of vorticity, which involves integrals of three-point vorticity characteristics. Multiply (1) with ω_j' and average (prime now indicates that field is taken at a point $\mathbf{x}'=\mathbf{x}+\mathbf{r}$). Symmetrization over (i,j) and use of homogeneity and isotropy gives

$$\frac{\partial\Omega_{ij}}{\partial t}+\frac{\partial\alpha_{ijk}}{\partial r_k}=2\nu\Delta\Omega_{ij}+Q_{ij},\qquad (10)$$

$$\Omega_{ij}(\mathbf{r})=\langle\omega_i\omega_j'\rangle,\ \ \alpha_{ijk}(\mathbf{r})=\langle\sigma_{jk}'\omega_i-\sigma_{ik}\omega_j'\rangle,\qquad (11)$$

$$\sigma_{ik}=v_k\omega_i-v_i\omega_k,\ \ Q_{ij}=2\langle\phi_i\omega_j'\rangle=\Phi_{ij}(\mathbf{r}).\qquad (12)$$

Here Ω_{ij} is the correlation tensor of vorticity. The tensor σ_{ik} is the vorticity flux, which combines convection and stretching of vortex filaments. The statistical mean value $\langle v_k\omega_i\rangle$ was called the "vorticity transport" tensor and was studied for a general nonhomogeneous turbulent flow [11]. The tensor α_{ijk} represents self-induced generation of vorticity correlations due to combined convection and stretching effects. Because the velocity field is an integral over the vorticity field, the tensor α_{ijk} is an integral over the three-point correlations of vorticity. The tensor Q_{ij} represents the influence of large-scale motion and we used (4) to evaluate this tensor.

The tensor α_{ijk} (11) is a sum of four moments of third order, which involve a product of two components of vorticity at different points and one component of velocity at one of these points. For isotropic turbulence we have the general expression

$$\langle v_i\omega_k\omega_j'\rangle=A\delta_{ik}n_j+B\delta_{ij}n_k+C\delta_{kj}n_i+Dn_in_jn_k,\qquad (13)$$

where $n_i=r_ir^{-1}$ and scalars $A,\dots,D$ depend on r. The condition of solenoidality

$$\frac{\partial}{\partial r_j}\langle v_i\omega_k\omega_j'\rangle=0\qquad (14)$$

gives

$$rA'+2A+B+C=0,\qquad\qquad (15)$$
$$rB'-B+rC'-C+rD'+2D=0$$

(prime indicates differentiation). Thus, only two scalars, say B and C, are independent. From (11)-(13) we obtain an expression with only one scalar:

$$\alpha_{ijk}(\mathbf{r})=\alpha(r)(2\delta_{ij}n_k-\delta_{ik}n_j-\delta_{jk}n_i),\ \ \alpha\equiv B-C.\qquad (16)$$

Here we used antisymmetry of tensor (13) with respect to vector $\mathbf{r}$.

Consider the inertial range of scales

$$l_\nu=\nu^{3/4}\epsilon^{-1/4}=L\,\mathrm{Re}^{-3/4}\ll r\ll L,\qquad (17)$$

where l_ν is the Kolmogorov internal scale. The variability of l_ν, due to intermittency [12-14], is not essential for the following analysis. By using solenoidality (2) and (8) we obtain

$$Q_{ij}(\mathbf{r})=2\delta_{ij}\epsilon L^{-2},\ \ r\ll L.\qquad (18)$$

Let us evaluate the first term on the rhs of (10). By using the definition of vorticity (2), we have

$$\Delta\Omega_{ii}=-\Delta^2\langle v_iv_i'\rangle=\tfrac{1}{2}\Delta^2\langle(v_i'-v_i)^2\rangle.\qquad (19)$$

Now we can use the "2/3 law" [13] (neglecting small intermittency correction):

$$\langle(v_i'-v_i)^2\rangle=\tfrac{11}{3}c_0(\epsilon r)^{2/3},\ \ c_0\approx 2.\qquad (20)$$

Here we indicated the empirical value of constant c_0 for the longitudinal structural function (which corresponds to the projection of velocity increment on $\mathbf{r}$). Substitution of (20) into (19) gives

$$\Delta\Omega_{ij}=a\epsilon^{2/3}r^{-10/3},\ \ a=\tfrac{220}{243}c_0\approx 1.8.\qquad (21)$$

By using isotropy and solenoidality, we finally get

$$\Delta\Omega_{ij}(\mathbf{r}) = 2a\epsilon^{2/3}r^{-10/3}(\delta_{ij} - \tfrac{5}{2}n_i n_j). \qquad (22)$$

For statistically stationary turbulence, substitution of (16), (18), and (22) into tensor equation (10) gives two equations for the scalar $\alpha(r)$:

$$r\alpha' + \alpha = 2a\nu\epsilon^{2/3}r^{-7/3} + \epsilon L^{-2}r, \qquad (23)$$

$$r\alpha' - \alpha = 5a\nu\epsilon^{2/3}r^{-7/3}. \qquad (24)$$

It is easy to see that these two equations are consistent and give a unique solution (without arbitrary constant):

$$\alpha(r) = -\tfrac{3}{2}a\nu\epsilon^{2/3}r^{-7/3} + \tfrac{1}{2}\epsilon L^{-2}r. \qquad (25)$$

This equation represents balance between self-induced generation of vorticity correlations, viscous diffusion, and influence of large-scale motion. When we approach the internal scale l_ν, Eqs. (10) and (25) turn into the balance of enstrophy and, as was shown above, the effect of large-scale motion for large Re is negligible: $\sim \mathrm{Re}^{-3/2}$. We recover this result by comparing two terms in the rhs of (25) for $r \sim l_\nu$. However, within the inertial range (17) these two terms become comparable (and compensate each other) at the scale

$$l_s = (3a)^{3/10} L\,\mathrm{Re}^{-3/10} \approx 1.66 L\,\mathrm{Re}^{-3/10}. \qquad (26)$$

This result shows that dynamics of vortical structures in the inertial range is not a simple cascade process, but involves intermediate characteristic scale. It seems that l_s is a natural length scale for the vortex strings indicated above. A previous candidate for this role was Taylor's microscale $\lambda \sim L\,\mathrm{Re}^{-1/2}$. However, from the present analysis we conclude that scale λ does not appear in the dynamics of vortical correlations. For moderate Re, the difference between these two scales is not big: $l_s/\lambda \sim \mathrm{Re}^{1/5}$. Detailed observations for larger Re, numerical experiments, and further analysis are needed in order to understand more deeply the dynamics and statistics of vortex structures with characteristic scale l_s. The time of formation of vortex strings (until they became unstable) can be estimated as $\tau_s \sim l_s(\epsilon l_s)^{-1/3} \sim T\,\mathrm{Re}^{-1/5}$, where $T \sim L^{2/3}\epsilon^{-1/3}$ is the external time scale.

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