Short-Time Behavior of Unstable Systems in Field Theory and Proton Decay

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(Received 23 March 1993)

The short-time behavior of an unstable particle is examined in a realistic field theoretical model. The onset of linear decreasing of the nondecay probability is shown to be extremely rapid, so rapid as to exclude any relevance of regimes quadratic in time ("Zeno" quantum paradox). The result applies to super-renormalizable, renormalizable, and nonrenormalizable cases, particularly to the proton decay problem. We discuss also the deviations from the exponential decay law in small-Q-value decays.

PACS numbers: 11.10.St, 13.30.Ce

Most studies in field theory deal with the asymptotic properties of microscopic processes, which can be reduced to the study of the S matrix. It is, however, interesting or in some cases necessary to study the finite-time evolution of quantum systems [1]. One such case is the behavior of the nondecay amplitude of an unstable system, where sizable deviations from the exponential behavior are expected to occur at short times [2].

Previous work on the problem has started from the nondecay probability at time T, $P_{\rm ND}(T)$, and its formal expansion at small times:

$$\begin{aligned} P_{\rm ND}(T) &= |\langle P|e^{-iHT}|P\rangle|^2 \\ &= 1 - T^2(\langle P|H^2|P\rangle - |\langle P|H|P\rangle|^2) + \cdots \\ &= 1 - \Delta E^2 T^2 + \cdots . \end{aligned} \tag{1}$$

According to Eq. (1), the decay rate vanishes for $T \to 0$, which leads to the so-called quantum Zeno paradox: The decay probability being negligible at small times, a quantum system observed frequently after formation will never decay.

In the case of proton decay, these considerations raise the suspicion that present experiments have failed to observe decay processes simply because protons in the Universe, some 10^{+10} years after formation, i.e., at times which are an infinitesimal fraction of the theoretical lifetime, are still in the quadratic regime, Eq. (1). If that were the case, the present experimental results would have no bearing on the issue of the proton's instability.

Several authors have addressed this problem, trying to recover the usual exponential decay law for proton decay at the present time. Some have considered the problem from the point of view of the theory of quantum measurements, which led to subtle questions as to whether the nuclear interactions inside the nucleus can or cannot be considered a quantum measurement of the proton nondecay probability [3]. Others have called into play the macroscopic nature of the measuring devices [4], or the macroscopic number of protons in the observed sample [5].

In this Letter, we consider the finite-time behavior of an unstable, field-theoretical system within perturbation theory (probably not inadequate for the problem of proton decay).

The first result that we find is that ΔE^2 in Eq. (1) is infinite in most cases. It is so even in the super-renormalizable case of a scalar theory with trilinear coupling, where we find

$$\Delta E^2 = \frac{\lambda^2}{8\pi^2 M}\int_0^\infty d\omega . \tag{2}$$

As a consequence, we find that Eq. (1) is useless in the most interesting cases and that the behavior at very short times depends critically on the details of the switching on, i.e., the formation of the unstable system.

This conclusion is very general and does not depend on the use of perturbation theory. In fact, the infinity of the first moment of the energy spectrum is not due to the choice of a particular interaction Hamiltonian or to the lack of a form factor. This infinity would be present also in case we decided to treat the proton as a composite QCD bound state. The divergence of the first moment is very generally due to the singular nature of the product of local observables (in this case the Hamiltonian density) when computed at short distances. As we show in the present paper, this divergence is tamed by eliminating the sharp observation time boundaries and introducing times characteristic of the measuring apparatus and independent of the physical decaying system. These rounding off parameters have obviously nothing to do with the extremely weak interaction determining the slow decay. Coming to the way the exponential decay is approached, we find that the relevant time scale in all cases considered and for "normal" Q values, $Q \simeq M$, is given by M^{-1} (M being the mass of the unstable system), with a possible enhancement proportional to a power of $(M\tau)^{-1}$, τ being the time scale characterizing the switching-on of the unstable system to which we have previously alluded. In the proton case, we expect τ to be of the order of the time scale of proton formation from the deconfined quark-gluon plasma in the early Universe, i.e., to be a typical strong-interaction time, like $(M_P)^{-1}$. Thus, a single proton is expected to enter the exponential decay region after an unobservably small time, of the order of 10^{-23} sec. Genuine deviations from the exponential law are present, instead, in decays with exceedingly small Q values, at times of the order $Q^{-1} \gg M^{-1}$, as observed in Ref. [6]

0031-9007/93/71(17)/2687(4)$06.00

and illustrated numerically at the end of this paper. Such effects have been searched for [7] but not yet observed.

We focus on the short-time behavior of simple field-theoretical unstable systems, such as a quantum field ψ of mass M, weakly coupled to a pair of lighter fields ϕ_1 and ϕ_2, so that the decay

$$\psi \to \phi_1 + \phi_2 \tag{3}$$

is allowed. We consider explicitly the following cases.

(i) Super-renormalizable interaction, with all scalar fields and a nonderivative coupling:

$$L_{\text{int}} = \lambda \psi \phi_1 \phi_2 + \text{counterterms}\,. \tag{4i}$$

(ii) Renormalizable interaction, with ψ and ϕ_1 spinor fields, Yukawa coupled to the scalar ϕ_2:

$$L_{\text{int}} = g\bar{\phi}_1 \psi \phi_2 + \text{H.c.} + \text{counterterms}\,. \tag{4ii}$$

(iii) Nonrenormalizable interaction; the same as before but with a derivative coupling of ϕ_2:

$$L_{\text{int}} = f(\bar{\phi}_1 \gamma_\mu \psi)\partial^\mu \phi_2 + \text{H.c.} + \text{counterterms}\,. \tag{4iii}$$

The nondecay probability at time T is defined according to Eq. (1). Going from the Schrödinger to the interaction representation, one finds

$$\begin{aligned} P_{\text{ND}}(T) &= |\langle P|e^{-iHT}|P\rangle|^2 = |\langle P,t=0|P,T\rangle_S|^2 \\ &= |\langle P,t=0|e^{-iH_0T}|P,T\rangle_I|^2 \\ &= |\langle P,t=0|P,T\rangle_I|^2 \\ &= \left|\langle P|T\left[\exp\left(-i\int_0^T H_I(t)dt\right)\right]|P\rangle\right|^2, \end{aligned} \tag{5}$$

where H is the total Hamiltonian, H_I the interaction Hamiltonian in the interaction picture,

$$\begin{aligned} &H = H_0 + H_I\,, \\ &H_I(t) = e^{iH_0t}H_Ie^{-iH_0t} = \int H_I(x,t)d^3x\,, \end{aligned} \tag{6}$$

the symbol $T[\cdots]$ denotes time ordering and we have used the fact that the state $|P\rangle$ is an eigenstate of H_0.

Equations (1) and (5) contain sharp time boundaries at $t=0$ and $t=T$. As we will see, this leads to the singular behavior of P_{ND}, illustrated by Eq. (2). The divergence is not cured by renormalization. Mass renormalization in lowest order (as well as wave-function and coupling constant renormalization, in the renormalizable case) occurs in the imaginary part of the nondecay amplitude and therefore it gives a λ^4 contribution to P_{ND}, as discussed below. The fact that ΔE^2 is divergent in the renormalized theory may be surprising at first sight. However, it is simply a reflection of the sharpness of the time boundaries, similar, in this respect, to the short-distance singularity one encounters in current-current correlation functions in renormalized field theory.

The sharp boundary is unrealistic both because the measuring apparatus cannot provide infinite frequencies, and because the formation of the unstable state requires a characteristic time. To account for this effect, we introduce a smoothed version of the Heaviside function:

$$\Theta_\tau(t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega\, e^{i\omega t}\frac{1}{\omega - i\epsilon}\tilde{\delta}_\tau(\omega)\,, \tag{7}$$

where $\tilde{\delta}_\tau(\omega)$ is the Fourier transform of some regularized version of the δ function. In the following, we use

$$\tilde{\delta}_\tau(\omega) = \frac{1}{1+(\omega\tau)^2}\,, \tag{8}$$

where, for simplicity, we have introduced a single characteristic time τ to describe both the switching-on and the switching-off of the measurement. Defining further,

$$g(t,T|\tau) = \Theta_\tau(t) - \Theta_\tau(t-T)\,, \tag{9}$$

the nondecay probability is written as

$$P_{\text{ND}}(T|\tau) = \left|\langle P|T\exp\left[-i\int g(t,T|\tau)H_I(t)dt\right]|P\rangle\right|^2. \tag{10}$$

In lowest order of perturbation theory, the corresponding nondecay amplitude, A_{ND}, can be written as

$$A_{\text{ND}}(T|\tau) = \langle P|T\exp\left[-i\int dt\, g(t,T|\tau)H_I(t)\right]|P\rangle = \text{Re}(A_{\text{ND}}) + i\,\text{Im}(A_{\text{ND}})\,, \tag{11}$$

with

$$\text{Re}(A_{\text{ND}}) = 1 - \frac{1}{2}\langle P|\int\int dt\,dt'\,g(t,T|\tau)g(t',T|\tau)H_I(t)H_I(t')|P\rangle\,, \tag{12i}$$

$$\text{Im}(A_{\text{ND}}) = -\langle P|\int dt\,g(t,T|\tau)H_I(t)|P\rangle - \frac{1}{2}\langle P|\int\int dt\,dt'\,\epsilon(t-t')g(t,T|\tau)g(t',T|\tau)H_I(t)H_I(t')|P\rangle\,. \tag{12ii}$$

We note the following features, valid at this order of perturbation theory.

(i) The usual ultraviolet (UV) divergences arise in the second term of $\text{Im}(A_{\text{ND}})$. The smoothing implied by the time scale τ has no effect on UV singularities. They are due to the locality of the T product, i.e., to the sharp behavior of the function $\epsilon(t-t')$, and can be regularized only by a genuine UV cutoff.

(ii) Terms in H_I of the first order in the coupling constant are nondiagonal and do not contribute to the first term of Eq. (12ii). Diagonal renormalization counterterms, needed to compensate the UV divergences, appear in the first term of $\text{Im}(A_{\text{ND}})$. They are at least second order in the coupling constant and give a higher order contribution to $P_{\text{ND}}(T|\tau)$.

In conclusion, we obtain

$$P_{\text{ND}}(T|\tau) = 1 - \langle P|\int dt\,dt'\,g(t,T|\tau)g(t',T|\tau)H_I(t)H_I(t')|P\rangle\,. \tag{13}$$

Inserting a complete set of intermediate states we get

$$P_{\rm ND}(T|\tau)=1-\int\int dt\,dt'g(t,T|\tau)g(t',T|\tau)\sum_n|\langle P|H_I(0)|n\rangle|^2e^{i(M-E_n)t}e^{-i(M-E_n)t'}$$
$$=1-\sum_n|\langle P|H_I(0)|n\rangle|^2\frac{4\sin^2[(2\omega-M)T/2]}{(2\omega-M)^2}[\tilde{\delta}_\tau(2\omega-M)]^2\,. \qquad (14)$$

Equation (14) does not yet give the physical survival probability of the particle. Among the intermediate states appearing in Eq. (14), there is a class corresponding to the disconnected diagrams shown in Fig. 1, where H_I excites and reabsorbs an intermediate state from the vacuum. These diagrams give rise to the nondecay amplitude of the bare vacuum, as indicated by the fact that they give an amplitude proportional to the space volume. Correspondingly, Eq. (14) factorizes according to

$$P_{\rm ND}(T|\tau)=P_{\rm vac}(T|\tau)P_{\rm part}(T|\tau)\,. \qquad (15)$$

$P_{\rm part}$ gives the survival probability of the particle, and it is also given by Eq. (14) but with the sum restricted to the contribution of connected diagram. Using the relation

$$\left[\frac{2\sin[(T/2)x]}{x}\right]^2\to_{(T\to\infty)}2\pi T\delta(x) \qquad (16)$$

we obtain

$$P_{\rm part}(T|\tau)\to_{(T\to\infty)}1-T\sum_n 2\pi\delta(2\omega-M)|\langle P|H_I(0)|n\rangle|^2=1-\Gamma T\to e^{-\Gamma T}\,. \qquad (17)$$

For $T\to\infty$, Eq. (17) reproduces the exponential decay, with the width Γ given by the Fermi "golden rule." Provided we choose $\tilde{\delta}_\tau(0)=1$, the limit is obtained independently from the value of the time scale τ. On the other hand, if we develop formally Eq. (14) around $T=0$ and set $\tilde{\delta}_\tau(x)=1$, we obtain a divergent result, i.e., Eq. (2) for the super-renormalizable interaction and a correspondingly higher divergence in the other cases.

What is relevant for us is the time scale at which the asymptotic limit (17) is attained. Of course, the intermediate-time behavior depends both on the particular form of the interaction and on the function $g(t,T|\tau)$. In the cases we are considering, Eqs. (2i)-(2iii), and in lowest order of perturbation theory, the contributions to Eq. (14) correspond to the direct and Z diagrams reported in Figs. 2(a) and 2(b). For illustration, we consider first the case of massless final particles. We find for the three different interactions

$$P_{\rm part,SR}=1-\frac{\lambda^2}{4\pi^2M}[F_0(T,M,\tau)+F_0(T,-M,\tau)]\,, \qquad (18\text{i})$$

$$P_{\rm part,R}=1-\frac{g^2M}{4\pi^2}[F_1(T,M,\tau)-F_1(T,-M,\tau)]\,, \qquad (18\text{ii})$$

$$P_{\rm part,NR}=1-\frac{f^2M^3}{4\pi^2}[F_3(T,M,\tau)-F_3(T,-M,\tau)]\,, \qquad (18\text{iii})$$

with

$$F_a(T,M,\tau)=\int_0^\infty d\omega\left(\frac{2\omega}{M}\right)^a\frac{\sin^2[(2\omega-M)T/2]}{(2\omega-M)^2}\times[\tilde{\delta}_\tau(2\omega-M)]^2\,. \qquad (19)$$

By explicit calculation, one finds

$$F_0(T,M,\tau)+F_0(T,-M,\tau)=\frac{1}{2}g_0\,, \qquad (20\text{i})$$

$$F_1(T,M,\tau)-F_1(T,-M,\tau)=\frac{1}{2}g_0\,, \qquad (20\text{ii})$$

$$F_3(T,M,\tau)-F_3(T,-M,\tau)=\frac{1}{2}\left(g_0+\frac{3}{M^2}g_2\right), \qquad (20\text{iii})$$

$$g_0=\frac{\pi}{2}T\left[1-\frac{3\tau}{2T}+\frac{1}{2}e^{-T/\tau}\left(1+\frac{3\tau}{T}\right)\right], \qquad (21)$$

$$g_2=\frac{\pi}{4\tau}\left[1-e^{-T/\tau}\left(1+\frac{T}{\tau}\right)\right]. \qquad (22)$$

The small-time behavior is then

$$P_{\rm part,SR}=1-\frac{\lambda^2}{64\pi M\tau}T^2+\cdots\,, \qquad (23\text{i})$$

$$P_{\rm part,R}=1-\frac{g^2M}{64\pi\tau}T^2+\cdots\,, \qquad (23\text{ii})$$

$$P_{\rm part,NR}=1-\frac{f^2M}{64\pi\tau^3}(3+M^2\tau^2)T^2+\cdots\,. \qquad (23\text{iii})$$

For large times, $MT\gg 1$ (but still such that $\Gamma T\ll 1$):

$$P_{\rm part,SR}=1-\frac{\lambda^2}{16\pi M}T\left(1-\frac{3\tau}{2T}+\cdots\right), \qquad (24\text{i})$$

$$P_{\rm part,R}=1-\frac{g^2M}{16\pi}T\left(1-\frac{3\tau}{2T}+\cdots\right), \qquad (24\text{ii})$$

$$P_{\rm part,NR}=1-\frac{f^2M^3}{16\pi}T\left[1+\left(\frac{3}{2M\tau}-\frac{3M\tau}{2}\right)\frac{1}{MT}+\cdots\right]. \qquad (24\text{iii})$$

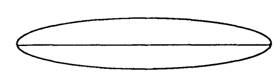

FIG. 1. Disconnected contribution to Eq. (14).

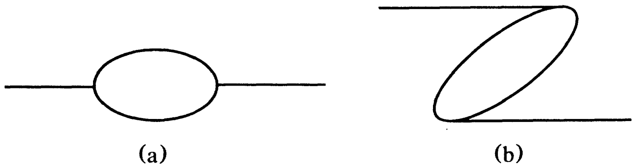

FIG. 2. (a) Direct diagrams and (b) Z diagrams contributing to Eq. (14).

A few comments are in order.

(i) Equations (23) show the fallacy of the finite ΔE^2 assumption. This phenomenon is an unavoidable consequence of the singular nature of the product of local observables (in this case the Hamiltonian density) when computed at short distances, as Eq. (13) clearly shows. In strict analogy to the theory of inclusive processes, the presence of form factors cannot eliminate this divergence and, as already stressed in the introduction, this singular behavior would also be present in case we decided to treat the proton as a composite QCD bound state. In particular, for very short rising times, $\tau \to 0$, the exact nondecay probability in the super-renormalizable case decreases linearly from unity, at all times; see Eq. (21).

(ii) The large-time behavior, Eqs. (24), indicates that the linearized exponential decay law, Eq. (17), is reached after characteristic time scales of the order of M^{-1}, enhanced by a factor proportional to $(M\tau)^{-1}$ in the nonrenormalizable case.

In the interesting case of muon decay, even assuming

$$(\tau M_\mu)^{-1} \simeq \frac{M_W}{M_\mu} \simeq 10^3 ,$$

an enhancement of the order of $(\tau M_\mu)^{-2}$ (appropriate for the four-fermion interaction) would give a time scale of the order of 10^{-17} sec, much smaller than the observable laboratory time scales.

In proton decay, as remarked above, expected formation time is itself in the order of $(M_P)^{-1}$, and no deviation from the exponential decay law is implied at the present time.

(iii) The conclusions in (ii) remain qualitatively unchanged for decays with massive final particles, provided the Q value of the decay is a sizable fraction of M, as shown in Fig. 3(a).

Genuine and perhaps observable deviations from the exponential decay law at small time scale are positively predicted [6] for decays with a very small Q value. For $Q \to 0$, there is a new time scale Q^{-1}, independent from and much larger than the natural time scale M^{-1}. The corresponding short-time deviation from the exponential law is illustrated in Fig. 3(b), for the super-renormalizable interaction.

Finally, we recall that the analysis presented here has no bearing upon the really large-time behavior, i.e., $\Gamma T \gg 1$, for which the perturbative approximation breaks down and a much more difficult analysis is required.

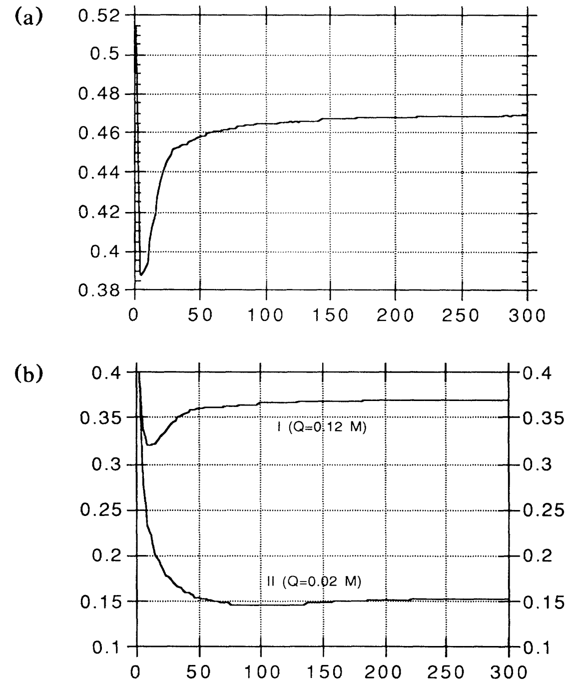

FIG. 3. (a) The function $(1-P_{\rm part})/T$ vs T, in units of M^{-1}. Super-renormalizable interaction, final particles of equal mass: $\mu=0.4M$. The switch-on and -off time is $\tau=(100M)^{-1}$. (b) Same as in (a), but for final particle mass $\mu=0.44M(I)$ and $\mu=0.49M(II)$. The increase of the characteristic time scale with Q^{-1} can be noticed.

Steps in this direction can be found in Ref. [2].

We thank A. De Gasperis, G. C. Ghirardi, and Y. Srivastava for useful discussions about previous work on the subject. This work was partially supported by the EEC Grant No. SCI-0430-C and by Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

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