Phase Slip and Phase-Slip Cascades in ^{4}He Superflow through a Small Orifice

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The behavior of quantized vortex filaments subject to flow through an idealized orifice is investigated. Small loops nucleated at the wall are entrained and stretched by the diverging flow of the orifice, leading to individual phase-slip events. A macroscopic remanent vortex, however, can repeatedly give rise to a cascade of phase slips draining large amounts of energy from the flow field.

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The onset of dissipation in superfluid ^{4}He is a problem which has been studied for many decades. Recently, a series of brilliant experiments [1-6] have shown that, when ^{4}He is made to flow through a micron-size orifice, dissipation first occurs in the form of discrete events, each of which takes an identical amount of energy from the applied flow field. The current interpretation of such events is that single, microscopically nucleated quantized vortex loops somehow interact with the applied flow so as to grow across the orifice, crossing all of the streamlines of the applied flow in the process. Such a "phase-slip" event changes the velocity potential difference across the orifice by the quantum of circulation κ, and must thus extract energy from the flow field.

It is useful to divide this problem into two distinct parts. First, one needs to consider how microscopic vortex loops can in fact fluctuate into the system, either thermally or quantum mechanically. Although this issue has received most of the attention, the lack of a microscopic theory of the ^{4}He superfluid state makes it a problematic one. Second, one needs to understand how a vortex loop interacts with the flow field to produce the observed behavior. Here one is on firmer ground, since the last ten years have taught us how to treat this kind of problem using classical vortex dynamics [7]. As it happens, the orifice experiments exhibit other types of behavior, such as multiple phase slips, and phase-slip *cascades* [3,8,9] which remove a large amount of flow energy in what is apparently a single dynamical event. These cannot be explained by the loop-nucleation ansatz, making it all the more intriguing to investigate the general features of vortex-filament dynamics in an orifice.

The orifices used in the actual experiments are of a complicated and only roughly characterized geometry. On the assumption that much of the observed behavior is generic to orifice flow, however, we have chosen to study an idealized orifice consisting of surfaces of constant η in the oblate spheroidal coordinate system

$$x = a\cos\phi[(\xi^2+1)(1-\eta^2)]^{1/2}, \tag{1a}$$

$$y = a\sin\phi[(\xi^2+1)(1-\eta^2)]^{1/2}, \tag{1b}$$

$$z = a\xi\eta. \tag{1c}$$

For any particular sheet $|\eta| = \eta_0$, the orifice is a hyperboloid of revolution (see Fig. 1), with an inner opening of radius $a(1-\eta_0^2)^{1/2}$. This geometry has the advantages that the applied flow through the orifice is known [10],

$$\mathbf{v}_a = \pm\frac{v(0,0,0)}{[(\xi^2+\eta^2)(\xi^2+1)]^{1/2}}\hat{\xi}, \tag{2}$$

and that one can make the edges of the hole as singular as one pleases by going to the limit $\eta_0 \to 0$. The motion $\dot{\mathbf{s}}$ of the vortex filament is determined by the condition that the force $\mathbf{f}$ exerted by the vortex on the fluid must equal the frictional force exerted by the excitation gas on the vortex core. This leads to the familiar equation [7],

$$\dot{\mathbf{s}} - \mathbf{v}_a + \beta\mathbf{s}'\times\mathbf{s}'' + \alpha\mathbf{s}'\times(\mathbf{v}_n - \mathbf{v}_a - \beta\mathbf{s}'\times\mathbf{s}''), \tag{3}$$

where the primes denote the derivative of the vortex curve $\mathbf{s}(\xi,t)$ with respect to the arc length, $\beta = -(\kappa/4\pi)\times\ln(|s''|a_0)$ is a logarithmic cutoff parameter involving the quantum of circulation κ and the vortex core radius a_0, and the term in α describes the frictional effect on the vortex of the excitation gas moving with an average ve-

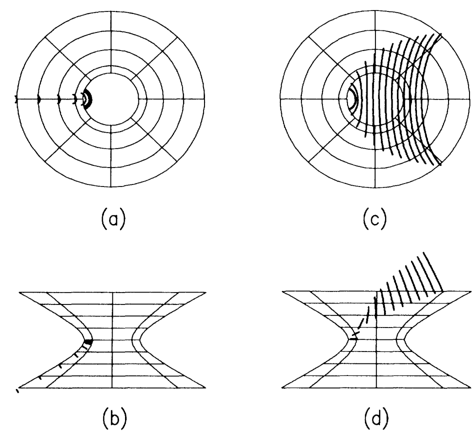

FIG. 1. Two ways in which a loop placed in an orifice can develop. (a) and (b) show the top and side view of the development of a loop 0.99 times the neutral size placed in the symmetry plane, while (c) and (d) show the behavior of a loop 1.01 times neutral size. The flow is through the orifice in the $\hat{\mathbf{z}}$ direction. For this calculation, $v_0(0,0,0)=2$, $\eta_0=0.5$, and $\alpha=0.01$. The reduced time interval between vortex configurations is 0.10 in (a) and (b), and 0.20 in (c) and (d).

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locity $\mathbf{v}_n$. In considering flow through a very small orifice, $\mathbf{v}_n$ can be assumed to be zero. All calculations described here were done in terms of reduced velocities $v_0=v/\beta$ and times $t_0=\beta t$, with the focal radius a in Eqs. (1) set to 1. Using the scaling properties [11] of Eq. (3) and ignoring a minor logarithmic correction, the corresponding velocities and times are $\beta v_0/a$ and $a^2 t_0/\beta$ for a hole with focal radius a.

Equation (3) is the motion that a curved vortex induces locally on itself. Corrections to this equation arise from velocity fields contributed by the more remote parts of the vortex, by other vortices, and by additional fields which must be introduced in order to satisfy the condition $\mathbf{v}_s\cdot\hat{\mathbf{n}}=0$ at the solid boundaries. These corrections have been neglected because they are extremely tedious to calculate and typically contribute only a 10% to 20% correction to the self-induced motion. The exception to this is when a vortex approaches another vortex or a boundary very closely. The ensuing nonlocal development can be well modeled in terms of an instantaneous vortex-vortex or vortex-boundary reconnection. With this reconnection ansatz, Eq. (3) becomes reasonably accurate, while retaining all of the interesting physics, and is thus particularly well suited for investigations where conceptual insight rather than a high degree of numerical accuracy is the goal. These issues are discussed in full in Ref. [7], where a fully nonlocal treatment of a vortex in a complicated geometry is given. On the basis of this earlier work, the results obtained here are expected to be qualitatively robust, but to be subject to errors of order 20%. Such errors are unimportant at the current stage of the subject.

For the calculations presented here, the orifice is assumed to be locally smooth: The end of a vortex (which must enter the boundary perpendicularly) then slides along the surface with a velocity determined by reflecting the vortex in the boundary [7]. The cutoff parameter β in Eq. (3) has been treated as a constant, in accord with the idea that numerical accuracy is unimportant at this stage. Finally, the calculations have been carried out with a small but finite α. This makes the numerical algorithm used to integrate Eq. (3) absolutely stable, without affecting the motion significantly.

We first investigate the nature of the single phase slips, which presumably result from the amplification of microscopic vortices nucleated as tiny half loops growing from the boundary. Calculations of the evolution of such a vortex loop, done in the limit of small α, show the following behavior. We consider a tiny loop originally lying in the $z=0$ symmetry plane of the orifice (the velocities are greatest here, making it the most likely environment for nucleation). For given $v_a(0,0,0)$, a, and η_0, there exists a stationary loop configuration in this plane, determined by the condition $\mathbf{v}_a(\mathbf{r})+\beta\mathbf{s}'\times\mathbf{s}''=0$. That is, the vortex starts out normally from the boundary and curves back to it so that its self-induced velocity exactly cancels the local flow field along its entire length. This neutral configuration maximizes the energy, and thus corresponds to the peak of the free energy barrier for vortex nucleation. A loop which is smaller than neutral has a larger self-induced velocity $\beta\mathbf{s}'\times\mathbf{s}''$ and will therefore propagate to $-\infty$ as shown in Figs. 1(a) and 1(b). A loop larger than neutral, on the other hand, will be convected in the flow direction and enlarged by the diverging $\mathbf{v}_a$ field, as shown in Figs. 1(c) and 1(d). Eventually, its self-induced motion will carry it across all of the streamlines of $\mathbf{v}_a$, the ends of the vortex passing around the orifice as shown.

It is of interest to ask how "dissipation" can be observed in the limit of very small α, in which negligible dissipative forces are acting in the system. One can show on very general grounds [12] that the energy which must be supplied by external forces acting on the vortex in order to create a loop such as that shown in Fig. 1 is

$$E=\frac{1}{2}\rho\int v_v^2 dV+\rho\kappa\int_s \mathbf{v}_a\cdot d\mathbf{A}\,, \tag{4}$$

where S is any surface bounded by the loop, and where the velocity field $\mathbf{v}=\mathbf{v}_a+\mathbf{v}_v$ has been written as the sum of the applied flow field and the velocity field due to the vortex. The first term of this equation represents the kinetic energy stored in the vortex field, while the second term represents energy added to the applied flow field as the loop grows. The second term, which is just the mass flux of the applied flow through the loop, can be made negative by orienting the loop properly, so that as the loop grows it sucks energy out of the flow field. In the present instance, when the external forces acting on the vortex are negligible, E cannot change as the vortex evolves. However, if the flow is such as to *stretch* the vortex loop, the first term on the right of Eq. (4) will increase at the expense of the second, which must become more negative. Thus, kinetic energy will be taken out of the large-scale flow field and stored in the more localized vortex field. No real dissipation occurs [13], but a loss of flow energy equal to κ times the mass flux through the hole will nevertheless be observed every time a vortex loop undergoes the evolution of Figs. 1(c) and 1(d) and propagates to infinity. Since it requires vortex stretching, this "frictionless dissipation" mechanism is peculiar to diverging flows such as those found in an orifice.

An upper limit for the time necessary to complete a phase-slip event can be estimated from a simple argument, which will be presented elsewhere. For a slitlike orifice of short dimension d and long dimension l, the characteristic crossing time is approximately

$$t_c\simeq V_a\frac{l^2 d}{\pi\beta^2}\,, \tag{5}$$

where V_a is the typical value of the flow velocity through the slit. The increase of t_c with V_a reflects the fact that the loop grows to a larger size as V_a becomes bigger, decreasing the self-induced velocity which carries it across the orifice. For a circular orifice of radius r_0, one can write $d\simeq l\simeq 2r_0$. Even though Eq. (5) then becomes very approximate, the resulting estimate is in reasonable agreement with our calculations. If one assumes a critical velocity of 500 cm s^{-1}, the estimated crossing times for the slit geometries used in Refs. [1–6] are on the order of a millisecond. This is tantalizingly close to the best experimental time resolution quoted [4]. While the phase slips measured so far seem instantaneous, their temporal evolution may become accessible if the time resolution can be modestly improved, or if the phase slips can be observed in larger orifices [14].

Suppose now that instead of amplifying a small vortex loop, the flow washes one end of a preexisting, remanent vortex into the orifice, the other end of the vortex being pinned somewhere far away. We have investigated the behavior of such an object, and find the very interesting result that it settles into a cyclic motion, the end of the vortex traveling around the hole and the vortex ballooning outward as shown in Figs. 2(a) and 2(b). Once every revolution, the vortex touches the boundary and breaks off an independent loop which then propagates away, as shown in Figs. 2(c) and 2(d). The calculation shown in Fig. 2 is for a rather low velocity. A calculation with a velocity more characteristic of the experiments (Fig. 3) reveals that what one has here is a *vortex mill*, somewhat similar in spirit and appearance to the vortex mill previously proposed [15] as a mechanism for initiating and maintaining superfluid turbulence in a channel. The significant difference is that in the case of the channel, the outward growth of the vortex spiral is caused by the friction term (i.e., $\alpha \neq 0$), whereas here vortex stretching due to the diverging flow in the orifice is the primary mechanism.

It is now easy to see how phase-slip cascades can occur. Imagine a remanent vortex with one end pinned by surface roughness near the orifice. At some time the vortex breaks loose (perhaps because it is impacted by the propagating loop created by a previous phase-slip event) and is washed into the hole. It then becomes a vortex mill, generating a continuous stream of phase-slip events, each one of which extracts a unit of energy from the applied flow field. In the actual experiments, the flow field is provided by exciting a Helmholtz oscillation across the orifice. Thus the flow alternates in direction, albeit with a period long compared to t_c. When a simple nucleated phase slip occurs, one unit of energy is extracted and the amplitude of the oscillation decreases by a small amount. When the vortex mill goes into operation, however, it will quickly drain away energy until the flow velocity has been reduced to the point at which the active end of the vortex repins and the mill stops. The Helmholtz amplitude thus collapses more or less catastrophically, depending on the details of the situation. The elegant feature of this process is that the vortex which mediated the cascade remains behind, ready to break loose again and repeat its performance once the oscillation amplitude has recovered. Thus it requires only a single remanent vortex to have repeated cascades.

It is interesting to note the observations of Varoquaux, Meisel, and Avenel [3], who report that cascades are initiated only when the flow is in one particular direction in their orifice, and that the initiation of each cascade is immediately preceded by a simple phase slip occurring during the previous half cycle, when the flow is in the oppo-

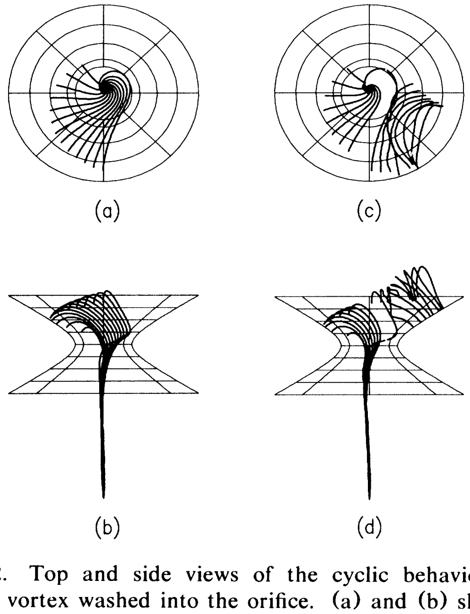

FIG. 2. Top and side views of the cyclic behavior of a remanent vortex washed into the orifice. (a) and (b) show the first half of the cycle. (c) and (d) start at the moment when the bulge which the vortex has developed touches the side of the orifice and reconnects, creating a loop that propagates away. The rotation of the vortex is counterclockwise in (a) and (c). The calculation was done with the same parameters as Fig. 1, the time step here being $\Delta t_0 = 0.20$.

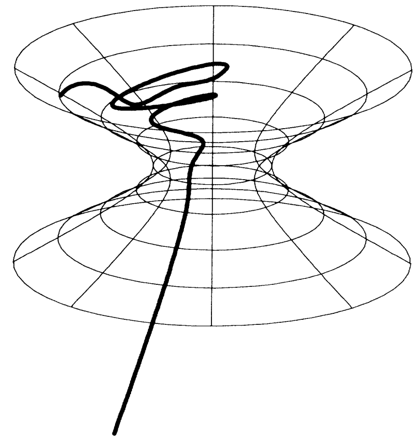

FIG. 3. Perspective view of the vortex mill when $v_0(0,0,0) = 8$ and the vortex comes in from off center. The motion is that of an outwardly growing spiral which periodically touches the boundary, creating a loop which then propagates away.

site direction. This is the behavior one would expect if only a single remanent vortex on one side of the orifice were involved, and if this vortex were broken loose by a collision with a vortex loop nucleated on the previous half cycle and now propagating away from the orifice. A second interesting point to be mentioned is that the cascade process and the single phase-slip process may be expected to scale differently with the size of the orifice. The nucleation process presumably depends primarily on the flow velocity at the surface, so that the critical velocity for the onset of single events should not depend very much on the hole size. The velocities describing the cascade process, on the other hand, should scale inversely as the hole size, if it is assumed that all features of the orifice are scaled in a corresponding manner. This implies, at least qualitatively, that the phase-slip cascades should dominate for large orifices, but become increasingly rare and eventually disappear altogether as the orifice is made smaller and the nucleated single phase-slip events dominate [16].

Although the complications arising from irregular hole geometries, time-dependent fields, and random loop nucleations still need to be factored in, it is possible to construct quite complex fluid dynamical behavior closely resembling that observed experimentally, using only the two basic mechanisms invoked here. In particular, the elucidation of the phase-slip cascade phenomenon as arising from the action of a single remanent vortex appears to be of value. It should now be possible to probe more complicated phenomena, such as multiple phase slips, the biasing of nucleation by nearby pinned vortices, and the transition to superfluid turbulence by carrying out more ambitious calculations in tandem with experiment.

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[12] A longer paper on this subject is in preparation. Some of the essential ideas were previously developed in E. R. Huggins, Phys. Rev. A **1**, 332 (1970) and K. W. Schwarz, Phys. Rev. B **12**, 3658 (1975).

[13] Eventually, the propagating vortex loop will of course dissipate, either by frictional decay or by annihilating on a boundary, but this will not affect the flow field.

[14] It should be noted that the actual crossing times may be shorter than predicted by Eq. (5), which is based on the assumption that the energy lost from the flow field goes into making the vortex larger as it crosses the orifice. If the orifice surface is locally rough, however, parts of the loop may annihilate as it crosses the orifice. Since small loops move more quickly than large loops, this reduces the crossing time. Such an effect is particularly likely to arise in the slit geometries of Refs. [1–6].

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[16] An illuminating conversation with Yu. Mukharsky on this point is acknowledged.