Measurements of the Probability Distribution of the Operationally Defined Quantum Phase Difference

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From a series of phase difference measurements by our scheme 2, in which the field at one interferometer input is phase shifted progressively in steps of 18°, we obtain the probability distribution of the phase difference. The results are in agreement with theoretical predictions based on our operational approach to the quantum phase.

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The well known problem of identifying dynamical variable(s) to represent the phase of a quantized electromagnetic field [1–4] has experienced a revival of interest in recent years [5–21]. Moreover, partly as the result of the work of Agarwal, Paul, Schleich, Vogel, and co-workers [21–25], attention is now focused not only on the operators and their expectations, but also on the entire probability distribution of the phase and on the quasiprobability functions that represent the density distribution of the state in phase space.

We have recently introduced an operational approach to the problem of identifying the quantum phase which is rooted in what is typically measured in classical optics rather than in fundamental theoretical considerations [26–29]. This approach leads naturally to the adoption of different operators for different measurement schemes, which is contrary to the point of view adopted in most other treatments. However, our approach to the phase problem has led to very good agreement with experiment, indeed to better agreement over a wider range of parameters than other theoretical treatments.

We focus on the measurement of the phase difference between two input fields, which we regard as more fundamental than the absolute phase, for reasons that were already pointed out a long time ago by Nieto [30]. But whereas the absolute phase is usually defined relative to a strong local oscillator, in weak fields the difference between the photon numbers counted in one measurement may be very small. For example, with one input to the measurement apparatus in the vacuum state and one input in a weak coherent state $|v\rangle$ with $|v|^2 \ll 1$, most measurements will result in zero photon counts and only infrequently is even one photon registered. Because the experimental outcome in which no photons are registered leaves the phase undefined, we have chosen to discard these outcomes and to renormalize the remaining probabilities accordingly. On the other hand, a one-photon outcome corresponds to one particular phase angle and to a δ-function probability density in our formalism, and this has been criticized [31,32]. In the following we show that, by a simple modification of our previously described measurement technique, it is possible to derive the entire (almost continuous) probability density of the phase difference. We also present new experimental results that confirm the theory.

Consider the 8-port experimental arrangement shown in Fig. 2 of Ref. [28], which has been designated measurement scheme 2 [26–29]. Incoming optical fields enter at input ports 1 and 2, where they are split into two. Two parts from ports 1 and 2 come together and interfere at beam splitter BS_3, while the other two parts interfere at beam splitter BS_5, after one beam traverses a $\lambda/4$ phase shifter. We can therefore simultaneously obtain the measured cosine C_M of the phase difference $\phi_2 - \phi_1$ from the photon counts registered by detector D_3, D_4 and the measured sine S_M of the phase difference from the photon counts registered by D_5, D_6, by use of the relations

$$\hat{C}_M(\phi_2 - \phi_1) = \frac{\hat{n}_4 - \hat{n}_3}{[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}}, \tag{1}$$

$$\hat{S}_M(\phi_2 - \phi_1) = \frac{\hat{n}_6 - \hat{n}_5}{[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}}. \tag{2}$$

Because $\hat{n}_4 - \hat{n}_3$ and $\hat{n}_6 - \hat{n}_5$ commute, there is no ambiguity in writing fractions as in Eqs. (1) and (2). Now the expectation of any operator function $f(\{\hat{n}\})$ of the set $\{\hat{n}\}$ of number operators is given by

$$\langle f(\{\hat{n}\})\rangle = \sum_{\{n\}} f(\{n\}) P(\{n\}), \tag{3}$$

where $P(\{n\})$ is the joint probability of $\{n\}$. $P(\{n\})$ may be expressed in terms of the input field operators $\hat{a}_1$, $\hat{a}_2$ as a normally ordered expectation [26]. We discard experimental results in which $n_4 = n_3$ and $n_6 = n_5$ because they do not lead to meaningful values of $\hat{C}_M$, $\hat{S}_M$, and this requires a renormalization by the factor $(1 - \mathcal{P}_0)$ giving the probability of not encountering 0/0 in Eqs. (1) or (2). We assume henceforth that this has been done. In particular, if we choose $f(\{\hat{n}\})$

$$\begin{aligned} f(\{\hat{n}\}) &= \left\{\frac{(\hat{n}_4 - \hat{n}_3) + i(\hat{n}_6 - \hat{n}_5)}{[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}}\right\}^x \\ &= [\hat{C}_M(\Delta\phi) + i\hat{S}_M(\Delta\phi)]^x \\ &\equiv e^{i\hat{\Theta}(\{n\})x} \end{aligned} \tag{4}$$

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in Eq. (3), $(\Delta\phi \equiv \phi_2 - \phi_1)$, then $\langle f(\{\hat{n}\})\rangle$ becomes the characteristic function $\mathcal{C}(x)$ of the phase difference $\Delta\phi$

$$\langle f(\{\hat{n}\})\rangle = \langle e^{i\hat{\Theta}x}\rangle \equiv \mathcal{C}(x), \tag{5}$$

and Fourier inversion yields the probability density of $\Delta\phi$,

$$p(\Delta\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{C}(x)e^{-ix(\Delta\phi)}\,dx. \tag{6}$$

Now suppose that input field 2 to the apparatus in Fig. 2 of Ref. [28] is phase shifted by an angle θ before it enters at port 2. Then the measured characteristic function becomes $\mathcal{C}(x;\theta) \equiv \langle \exp[i\hat{\Theta}x]\rangle'$, where $\langle\ \rangle'$ denotes the expectation in the phase shifted quantum state. The corresponding probability density of $\Delta\phi$, conditioned on the phase shift θ, is given by

$$p(\Delta\phi \mid \theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{C}(x;\theta)e^{-i(\Delta\phi-\theta)x}\,dx. \tag{7}$$

If we make repeated measurements for several different values of the phase shift θ in equal steps B, so that $\theta = rB$, where $r = 0, \pm1, \pm2, \ldots, \pm\pi/B$, and if we choose B so that π/B is an even integer, the desired operational probability density of the phase difference $P(\Delta\phi)$ can be obtained by averaging the values of $p(\Delta\phi \mid rB)$ over r. Thus

$$P(\Delta\phi) = \frac{1}{\frac{2\pi}{B}+1}\sum_{r=0}^{\pm\pi/B} p(\Delta\phi \mid rB). \tag{8}$$

Under these circumstances it is natural to present the results of the measurements as a histogram with bins of the same width B. Let the bins be labeled by $N = 0, \pm1, \pm2, \ldots, \pm\pi/B$. Then the probability P_N of encountering the phase difference $\Delta\phi$ in bin N is given by

$$P_N = \frac{1}{\frac{2\pi}{B}+1}\sum_{r=0}^{\pm\pi/B}\int_{(N-\frac{1}{2})B}^{(N+\frac{1}{2})B} p(\Delta\phi \mid rB)\,d(\Delta\phi)\,. \tag{9}$$

As B gets smaller and smaller P_N tends toward the continuous probability distribution $P(\Delta\phi)$. One advantage of working with the discrete histogram P_N rather than with $P(\Delta\phi)$ is that δ-function singularities of $P(\Delta\phi)$ are removed in the integration. This is illustrated by the following examples.

The so-called "split photon" input state, which was treated in Refs. [26,28], is a limiting case of the two-mode coherent product state $|v_1\rangle_1|v_2\rangle_2$ when $|v_1|, |v_2| \ll 1$. Then

$$|v_1\rangle_1|v_2\rangle_2 = |\text{vac}\rangle_{1,2} + v_1|1\rangle_1|\text{vac}\rangle_2 + v_2|\text{vac}\rangle_1|1\rangle_2 + \mathcal{O}(|v|^2), \tag{10}$$

and because the vacuum or all-zero photon contribution is automatically discarded in our measurements, the results for this state are just like those for the one-photon superposition state

$$|\Psi\rangle = \beta_1|1\rangle_1|0\rangle_2 + \beta_2|0\rangle_1|1\rangle_2,$$
$$\beta_j = v_j/(|v_1|^2 + |v_2|^2)^{1/2} \quad (j=1,2)\,. \tag{11}$$

The measured characteristic function $\mathcal{C}(x;\theta)$ is easily evaluated in the state $|\Psi\rangle$ and, after Fourier inversion and on putting $\theta = rB$ and $\eta = \arg\beta_2 - \arg\beta_1$, we obtain for the average or operational probability density $P(\Delta\phi)$ from Eq. (8),

$$P(\Delta\phi) = \frac{1}{\frac{2\pi}{B}+1}\sum_{r=0}^{\pm\pi/B}\frac{1}{4}\{[1-2|\beta_1\beta_2|\cos(\eta+rB)]\delta(\Delta\phi+rB-\pi) + [1+2|\beta_1\beta_2|\cos(\eta+rB)]\delta(\Delta\phi+rB-2\pi) + [1-2|\beta_1\beta_2|\sin(\eta+rB)]\delta(\Delta\phi+rB-3\pi/2) + [1+2|\beta_1\beta_2|\sin(\eta+rB)]\delta(\Delta\phi+rB-\pi/2)\}\,. \tag{12}$$

This includes the effect of renormalization by the factor $(1-\mathcal{P}_0)$. The presence of δ functions simply reflects the fact that each photocount outcome corresponds to one particular value of the phase difference. From Eq. (9) the probability P_N that a given phase difference falls into bin N is given by

$$P_N = \left(\frac{1}{\frac{2\pi}{B}+1}\right)[1+2|\beta_1\beta_2|\cos(\eta-NB)], \quad -\frac{\pi}{B} \le N \le \frac{\pi}{B} \tag{13}$$

and this has no singularities. In particular, when $\beta_2 = 0$, so that the input at port 2 is the vacuum state, Eq. (13) yields a uniform probability distribution of the phase difference, as would be expected for a Fock state. The singularities that appear before binning, which were discussed previously [28], evidently disappear.

More generally, for an arbitrary two-mode coherent product state $|v_1\rangle_1|v_2\rangle_2$ we obtain for $P(\{n\})$ [Eq. (76) of Ref. [26]]:

$$P(\{n\}) = \frac{|v_1-v_2|^{2n_3}}{4^{n_3}n_3!}\frac{|v_1+v_2|^{2n_4}}{4^{n_4}n_4!}\frac{|-iv_1+v_2|^{2n_5}}{4^{n_5}n_5!}\frac{|-v_1+iv_2|^{2n_6}}{4^{n_6}n_6!}e^{-(|v_1|^2+|v_2|^2)}\,. \tag{14}$$

With the help of the foregoing procedure this gives $p(\Delta\phi \mid \theta = rB)$ and hence P_N. In the limit $B \to 0$ the continuous probability distribution $P(\Delta\phi)$ can be obtained directly by writing, with $\beta \equiv 2|v_1||v_2|/(|v_1|^2+|v_2|^2)$,

$$P(\Delta\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\Delta\phi \mid \theta)\,d\theta$$
$$= \frac{1}{2\pi}\left(\frac{1}{1-\mathcal{P}_0}\right)\sum_{\{n\}}(|v_1|^2+|v_2|^2)^{n_3+n_4+n_5+n_6}e^{-(|v_1|^2+|v_2|^2)}$$
$$\times\frac{[1-\beta\cos(\eta+\Theta-\Delta\phi)]^{n_3}}{4^{n_3}n_3!}\,\frac{[1+\beta\cos(\eta+\Theta-\Delta\phi)]^{n_4}}{4^{n_4}n_4!}$$
$$\times\frac{[1-\beta\sin(\eta+\Theta-\Delta\phi)]^{n_5}}{4^{n_5}n_5!}\,\frac{[1+\beta\sin(\eta+\Theta-\Delta\phi)]^{n_6}}{4^{n_6}n_6!} \tag{15}$$

and this continuous probability distribution is plotted in Fig. 1 for $v_1 = 1$, $v_2 = 2.36$. $\Theta(\{n\})$ is the function of $\{n\}$ defined by Eq. (4).

It is interesting to compare the form of $P(\Delta\phi)$ given by our theory with that given by another theory for the phase, such as that of Pegg and Barnett [10], for example. If $|\Phi_1\rangle_1|\Phi_2\rangle_2$ is the two-mode product phase state, with the phase state $|\Phi\rangle$ defined within the truncated Hilbert space [10] by

$$|\Phi\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s} e^{in\Phi}|n\rangle\,, \tag{16}$$

and with the understanding that we allow $s \to \infty$ at the end, we have for the probability density $p_{\mathrm{PB}}(\Delta\phi)$ of the phase difference $\Delta\phi$ given by Pegg and Barnett,

$$p_{\mathrm{PB}}(\Delta\phi) = \left(\frac{s+1}{2\pi}\right)^2\int_{-\pi}^{\pi}\left|\,{}_1\langle\Phi_1|{}_2\langle\Phi_1+\Delta\phi|v_2\rangle_2|v_1\rangle_1\right|^2 d\Phi_1\,. \tag{17}$$

Expansion of the coherent states and the phase states in terms of Fock states leads to the following result:

$$p_{\mathrm{PB}}(\Delta\phi) = \frac{1}{2\pi}[1+e^{-(|v_1|^2+|v_2|^2)}]\sum_{\substack{\{n\}\\ n_1\neq n_1',n_2\neq n_2'\\ n_1+n_2=n_1'+n_2'}}\cos[(n_1-n_1')(\Delta\phi-\eta)]\frac{|v_1|^{n_1+n_1'}|v_2|^{n_2+n_2'}}{(n_1!n_2!n_1'!n_2'!)^{1/2}}\,. \tag{18}$$

This probability density is also shown in Fig. 1 for the case $v_1 = 1$, $v_2 = 2.36$ for comparison with our theory. The curves clearly differ significantly. Because our theory for measurement scheme 2 incorporates the effects of the 50%:50% beam splitters BS_1 and BS_2, whereas the Pegg and Barnett theory does not, we also show in Fig. 1 the results given by Eq. (18) when the field amplitudes are rescaled so that $|v_1|^2 = 0.5$, $|v_2|^2 = 2.36^2/2 = 2.78$. The agreement with our theory is no better in this case than before. We therefore conclude that our phase measurement scheme is not well described by the Pegg and Barnett formalism.

We have tested these theoretical predictions experi-

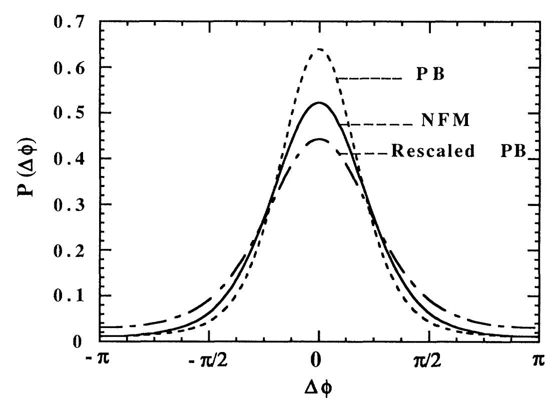

FIG. 1. Predicted probability distribution $P(\Delta\phi)$ of the phase difference $\Delta\phi$ between two input fields in the coherent state $|v_1\rangle|v_2\rangle$ with $v_1 = 1$, $v_2 = 2.36$. The full curve is based on Eq. (15) for our (NFM) theory. The two broken curves are based on Eq. (18) for the Pegg and Barnett theory, with and without rescaling.

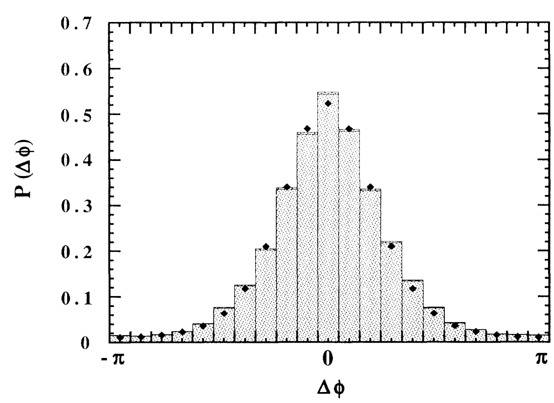

FIG. 2. The measured probability distribution $P(\Delta\phi)$ as a function of the phase difference $\Delta\phi$ for the two-mode coherent state $|v_1\rangle|v_2\rangle$ with $v_1 = 1$, $v_2 = 2.36$ and bin size $B = 18°$. The two horizontal lines at the top of each bin correspond to the mean ± 1 standard deviation. The black dots show the theoretically expected values given by our (NFM) theory.

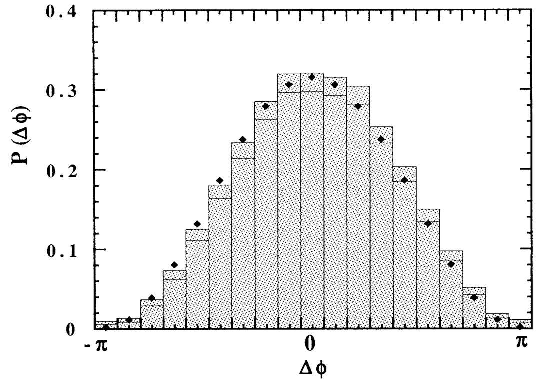

FIG. 3. The measured probability distribution $P(\Delta\phi)$ as a function of the phase difference $\Delta\phi$ for the "split photon" state $|\Psi\rangle$ given by Eq. (11) with $\beta_1/\beta_2 = 1.07$ and with bin size $B = 18°$. The two horizontal lines at the top of each bin correspond to the mean ± 1 standard deviation. The black dots show the theoretically expected values given by Eq. (13) for our (NFM) theory.

mentally with the apparatus shown in Fig. 2 of Ref. [28] and described in more detail in Refs. [27,28]. The input light beams were derived by splitting a very stable single-mode He:Ne laser beam into two beams, as before [27–29]. The differential phase shift $\theta = rB$ at the interferometer input was generated by piezoelectric displacement of the input beam splitter in phase steps of $B = 18°$. The photon numbers registered by the four equally sensitive detectors D_3, D_4, D_5, D_6, in a measurement interval of 200 ns, were recorded many thousand times.

Figure 2 shows the histogram of values resulting from measurements of the two-mode coherent input state $|v_1\rangle|v_2\rangle$ with $v_1 = 1$, $v_2 = 2.36$. Also shown superimposed are the theoretically expected values derived from our theory by use of Eq. (14). Figure 3 gives the corresponding experimental results for the "split photon" state $\beta_1|1\rangle_1|0\rangle_2 + \beta_2|0\rangle_1|1\rangle_2$ with $\beta_2/\beta_1 = 1.07$, together with theoretical values given by Eq. (13).

In both cases there is reasonable agreement between our theory and experiment. We suspect that the somewhat worse agreement in Fig. 3 is connected with slow drifts of the imposed phase shifts θ, because the significantly lower counting rates in Fig. 3 required a 12 times longer total measurement time. It is worth noting that the discarded measurement outcomes amount to no more than 2.5% of the total in Fig. 2, but to 98% of the total in Fig. 3.

It is apparent that our experimental procedure is not limited to giving the average sine or cosine of the phase difference $\Delta\phi$, but that it allows the probability distribution of $\Delta\phi$ to be obtained from the data. Moreover, the experimental results agree quite well with our operational theory of the phase. Finally, these experiments obviously do not test those theoretical approaches to the phase that give the probability distribution directly but require the use of a strong local oscillator at one interferometer input [21–25].

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