Must Time-Machine Construction Violate the Weak Energy Condition?

Amos Ori
Department of Physics, Technion-Israel Institute of Technology, Haifa, 32000, Israel
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We present a time-machine model in which closed timelike curves evolve, within a bounded region of space, from a well-behaved spacelike initial slice S; this slice (and the entire spacetime) is asymptotically flat and topologically trivial. In addition, this model satisfies the weak energy condition everywhere on S and up until and beyond the time slice (an achronal hypersurface) which displays the causality violation. We discuss the relation of this model to theorems by Tipler and Hawking which place constraints on time-machine solutions.

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In the last few years considerable attention has been given to the problem of causality violation due to the formation of closed timelike curves (CTCs). The main question is whether general relativity allows one to produce CTCs in our Universe (or in an asymptotically flat spacetime) if none exist *a priori*. Morris, Thorne, and Yurtsever [1] raised this question five years ago, suggesting an interesting model based on a traversable wormhole. Their model demonstrates the formation of CTCs at some particular moment and location in spacetime. Then, in 1991, Gott [2] suggested an even simpler model of a time machine, based on two infinitely long rapidly moving cosmic strings.

While these models are rather elegant, they both seem to suffer from fundamental difficulties. The first model is problematic because one needs a concentration of negative energy to maintain the wormhole (i.e., to prevent it from collapsing to a singularity). This is a problem because we expect the energy-momentum tensor $T_{\alpha\beta}$ of any realistic material to satisfy the *weak energy condition* (WEC), according to which $T_{\alpha\beta}u^{\alpha}u^{\beta}\geq 0$ for every timelike or null vector u^{α}. In other words, any physical observer is expected to measure non-negative energy density. All known forms of (classical) matter fields satisfy the WEC. (Under certain conditions, quantum fields may violate the WEC, but there are strong constraints on such a violation [3,4].) In addition, the formation of a wormhole involves a change of topology, which is by itself problematic [5,6]. In Gott's two-string model, the energy density is always non-negative and the topology is trivial; however, this model is not asymptotically flat, as the strings are infinitely long. Moreover, it turns out that in this model CTCs are running toward the "center" from infinity [7,8]—a situation which has little to do with the creation of a time machine by a human being.

One might hope to get rid of the above difficulties by improving the model. But Tipler [9,10] has proved a number of theorems which put strong constraints on such a possibility. In rough terms, these theorems state the following: If there exists an asymptotically flat partial Cauchy surface S, and a formation of CTCs occurs in some bounded spatial region to the future of S, and if the energy density is nowhere negative, then the spacetime must be singular (i.e., geodesically incomplete). In other words, if we want the solution to be nonsingular and to describe the formation of a time machine within some finite region of space from "normal" initial data, then the WEC must be violated. This relation between the formation of CTCs and the violation of the WEC has been recently strengthened by Hawking [6]. Hawking's analysis assumes a compactly generated time machine, i.e., a situation in which all the past-directed null generators of the Cauchy horizon enter some compact region C and stay there forever. Hawking then proved that if a compactly generated causality violation occurs to the future of a noncompact partial Cauchy surface, then the WEC must be violated somewhere. He further showed that in such a situation there exists an instability of quantum matter fields.

In this paper we shall ignore questions of stability, and focus attention on the relation between causality violation and the WEC. Thus, we shall *assume* that a realistic solution should satisfy the WEC, and ask the following question: Do the presently known constraints on causality violation completely rule out the possibility of constructing a time machine from materials which satisfy the WEC? We shall provide a solution which suggests that —at least to some extent— the answer is no.

The standard interpretation of Tipler's theorems is to say that the appearance of a singularity in a given model indicates that this model is unrealistic and cannot be physically realized: Even for future-generation engineers it will probably be impossible to use "singular matter" for the construction of their time machine [10]. However, the theory of black holes provides an obvious counterexample to this interpretation. For, by applying this interpretation to the black hole's singularity theorems one could conclude that black holes can never form. This analogy makes it clear that Tipler's theorems can bear a very different interpretation, namely, that the construction of a time machine is perhaps possible, but that the causality violation will then inevitably lead to the formation of a singularity. This possible interpretation suggests that one should discard a time-machine model due to a singularity only if this singularity appears sufficiently early that it can causally interfere with the occurrence of

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causality violation. In other words, we shall not worry too much about late singularities if a causality violation occurs already in $\overline{D^+(S)}$ [of course, $D^+(S)$ itself cannot include any closed causal curve]. Hereafter we are using the notation of Ref. [11].

A similar criterion should be applied to regions where the WEC is violated. Although we presume here that a realistic solution must satisfy the WEC, for mathematical convenience it will sometimes be useful to consider a solution in which the WEC is initially satisfied but is violated later. By assumption, the late parts of the spacetime (those parts which violate the WEC) are of no physical relevance, but the early parts could still be relevant. Consider a solution g in which the WEC is satisfied everywhere except in some region V (we assume $V \cap S = \Phi$). Assume that there exists some configuration of matter fields which yields an energy-momentum tensor that agrees with g everywhere except in V. Suppose that one arranges the initial data (for both gravitational and matter fields) on S accordingly. Then, at least initially, the evolving geometry will agree with g. At later stages, the evolving geometry will deviate from g—not only in V, but generally in the whole range $J^+(V)$. Nevertheless (assuming that the matter fields possess a well-posed initial value problem), by causality the evolving geometry must agree with g in the range $P \equiv D^+(S) - J^+(V)$. The whole region P (and, by continuity, $\bar{P}$, too) is thus physically meaningful. Note that the definition of P takes care of causal effects of both singularities and violations of the WEC. We may thus accept a time-machine solution, even if some part of it violates the WEC or is singular, if the causality violation occurs already in $\bar{P}$ (that is, if $\bar{P}$ includes closed causal curves).

We shall now give a solution in which a causality violation occurs in $\bar{P}$, within some bounded region of space, to the future of a partial Cauchy surface S which is asymptotically flat and topologically trivial. The solution depends on six parameters: $a > 0$, $b > 0$, $r_0 > 0$, $0 < d < r_0$, $k > 0$, and q. Our starting point is the Minkowski line element, which we express in polar coordinates

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 . \tag{1}$$

(Later we shall also use spherical coordinates $-dt^2 + dR^2 + R^2 d\Omega^2$ for Minkowski.) We define $\rho^2 \equiv (r - r_0)^2 + z^2$. Any surface $0 < \rho = \text{const} < r_0$ is a torus. We shall now modify the geometry within the torus $\rho = d$ by writing

$$dS^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + d\tilde{s}^2 , \tag{2}$$

where

$$d\tilde{s}^2 = 2rh\{atdt - b[(r - r_0)dr + zdz]\} d\varphi + r^2 h^2 [b^2 \rho^2 - a^2 t^2] d\varphi^2 , \tag{3}$$

and where

$$h = h(\rho) \equiv [1 - (\rho/d)^4]^3 \tag{4}$$

for $\rho < d$ and $h = 0$ otherwise. Thus, outside the torus $\rho = d$ the Minkowski geometry is untouched. In particular, the region $R > d + r_0$ remains Minkowski. Finally, we define the physical metric ds^2 by

$$ds^2 = F(t) dS^2 , \tag{5}$$

where

$$F(t) \equiv 1 + q(t - 1/a) - k(t - l/a)^2 . \tag{6}$$

Outside $\rho = d$ the geometry is conformally flat, describing a homogeneous and isotropic cosmology. Thus, as it stands, the metric is not asymptotically flat. However, because the geometry is spherically symmetric for $R > \rho + d$, it is relatively simple to introduce a cutoff at some $R_0 > \rho + d$ and to match the solution to Schwarzschild (either sharply or smoothly) at $R > R_0$—without violating the WEC. This will be shown in Ref. [12].

The main features of the above solution are represented in Fig. 1. Qualitatively, the way the CTCs form is similar to the axially symmetric time-machine solution described in Sec. II of Ref. [6]. The circular orbits (namely, the orbits $t = \text{const}$, $r = \text{const}$, $z = \text{const}$), which are always closed, are initially spacelike everywhere. The effect of the term $atdt$ in Eq. (3) is to tip the light cone in the $-\varphi$ direction. At the critical moment $t = 1/a$, the effect is so strong that the circular orbit $\rho = 0$ becomes null ($g_{\varphi\varphi} = 0$). It is easy to show that this closed null orbit is a (future-incomplete) null geodesic, and we denote this closed null geodesic by N. At $t > 1/a$, the circular orbits in the central parts of the torus $\rho = d$ are timelike—i.e., CTCs. Because of the finite support of h, the region of circular CTCs is limited to the interior of $\rho = d$. The parameters r_0 and d control the dimensions of this torus. The main qualitative difference between our solution and the one in Ref. [6] is that in the latter the WEC is violated along the closed null geodesic. Here, thanks to the parameter b (which expresses a focusing of the generators towards N) the WEC is satisfied at N for ap-

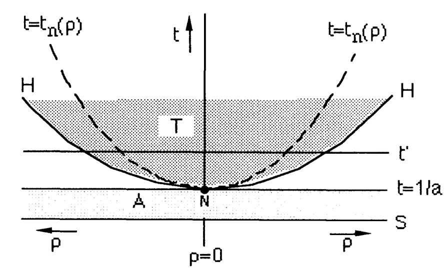

FIG. 1. A spacetime diagram showing a cut ($\varphi = \text{const}, z = 0$) through the time-machine spacetime. The region shown is the central part of the torus $\rho = d$, which forms the "core" of the time machine. The heavily dotted region T is the causality-violating set. The dot N (which is in fact a ring) is a closed null geodesic. In the lightly dotted region A all surfaces $t = \text{const}$ are spacelike. The WEC is satisfied in the whole region extending from the initial slice S to the hypersurface $t = t'$.

propriate choice of parameters. The conformal factor F has been introduced in order to prevent a violation of the WEC away from N.

We shall now look in more detail at some central features of the time-machine solution, Eqs. (1)-(6). Hereafter, we shall only be interested in the vicinity of $t=1/a$. First, $\det(g)=-F^4r^2$ nowhere vanishes in $\rho<d$. The metric and inverse metric components are well defined and C^∞ everywhere—except at $\rho=d$ where they are C^2. Consequently, the curvature and $T_{\alpha\beta}$ are well defined and continuous everywhere.

The hypersurfaces $t=$const (properly extended throughout the Schwarzschild region to spacelike infinity) are topologically trivial and asymptotically flat. A straightforward calculation yields $g^{tt}=F^{-1}[(hat)^2-1]$. Since $h\le 1$, $g^{tt}<0$ for every $0<t<1/a$. Therefore, any hypersurface $t=$const in this range is purely spacelike, and can serve as a partial Cauchy surface. Let S be such a hypersurface, with $0<t=t_s<1/a$. The region $t_s<t<1/a$ (denoted A in Fig. 1) is thus free of any causality violation. However, at $t>1/a$ a causality violation does occur. The metric component

$$g_{\varphi\varphi}=Fr^2[1+h^2(b^2\rho^2-a^2t^2)]\,,$$

which is always >0 in A, becomes negative at $t>1/a$ in the vicinity of $\rho=0$. Therefore, for any $t>1/a$, the circular orbits in the vicinity of $\rho=0$ are CTCs. The circular orbits are strictly null on the hypersurface $t=t_n(\rho)$ (see Fig. 1), where $t_n(\rho)$ is defined by

$$a^2t_n^2=b^2\rho^2+[1-(\rho/d)^4]^{-6}\,,$$

and they are timelike at $t>t_n(\rho)$. [For $\rho\ll(d,b^{-1})$ one gets $t_n\cong 1/a+(b^2/2a)\rho^2$.] One can easily show that, first, the causality-violation region T extends beyond the curve $t_n(\rho)$, and second, that the intersection of a hypersurface $t=\text{const}>1/a$ (e.g., the hypersurface t' in Fig. 1) with T is compact. That is, the causality violation occurs within a finite region of space.

In Ref. [12] we use the limit $r_0\to\infty$ (in which the geometry becomes cylindrically symmetric) to show the existence of combinations of parameters for which the WEC is satisfied at the critical moment $t=1/a$. Two such examples are

$$a=6d^{-1},\quad b=2d^{-1},\quad k=100d^{-2},\quad q=3d^{-1} \tag{7}$$

and

$$\begin{aligned}&a=100d^{-1},\quad b=10d^{-1},\\&k=5000d^{-2},\quad q=-40d^{-1}\,,\end{aligned} \tag{7$'$}$$

provided that r_0 is sufficiently large. (It turns out that if r_0 is too large, then trapped or antitrapped surfaces necessarily exist in the solution. Whether it is possible to avoid such trapped and antitrapped surfaces and yet to satisfy the WEC is still an open question.) Then, by continuity, for sufficiently small $\delta>0$ the metric ds^2 [Eqs. (1)-(6)] with the parameters (7) or (7′) satisfies the WEC in the whole range $1/a-\delta<t<1/a+\delta$. We choose the initial slice S and the slice $t'>1/a$ (see Fig. 1) to be within that range.

In Ref. [6] Hawking has shown that a compactly generated Cauchy horizon (which evolves from noncompact S) includes a closed null geodesic, and the WEC must be violated somewhere along this geodesic. But in our solution the closed null geodesic N nowhere violates the WEC. This shows that the geodesic N is not exactly of the type considered in Ref. [6]. And indeed, a close look at our metric reveals that the generators of the Cauchy horizon H focus onto N in the *future* direction, rather than the past. In other words, N is not the past "edge" of H; rather, it is the future edge of some section of H (Fig. 1 is somewhat misleading in this respect, because in T the hypersurfaces $t=$const are *not* everywhere spacelike). Yet, Fig. 1 makes it clear that a sequence of spacelike hypersurfaces ($t=\text{const}<1/a$) can approach N from the past without first intersecting H elsewhere.

It is likely (though not yet proved) that in our solution H is compactly generated. We therefore [6] expect the WEC to be violated somewhere. But this violation can only occur at $t>t'$. Since in the region $t<1/a$ all hypersurfaces $t=$const are spacelike, this region is causally isolated from any event at $t>1/a$. This ensures that the closed causal curve N belongs to $\bar{P}$.

Is it possible to modify the above solution in such a way as to completely avoid *any* singularity or violation of the WEC? We first point out that such a possibility does not seem to violate Tipler's theorems. In these theorems (as in most black hole's singularity theorems) the notion of singularity is defined by geodesic incompleteness. We are used to interpreting this incompleteness as an indicator for a physical curvature singularity. This interpretation is probably justified for black holes, but perhaps not for causality violation. For causality violation is often associated with the formation of an incomplete closed null geodesic [6]. This kind of incompleteness, known as *imprisoned incompleteness*, does not express any local irregularity in the geometry, and should not be regarded as a spacetime singularity [11]. The curve N in our solution is such an incomplete closed null geodesic (though not exactly of the type discussed in [6]). Thus, in Tipler's theorems the demands for geodesic incompleteness may be well satisfied by N, without any true spacetime singularity.

A stronger constraint on solutions which everywhere satisfy the WEC emerges from Hawking's analysis [6], which shows that such solutions cannot have a compactly generated Cauchy horizon. This raises the question as to what extent the criterion of compact generation is required by physical grounds. The main goal of this criterion is to rule out solutions like Gott's spacetime [2], in which the violation of causality begins at infinity [7,8]. Intuitively, one would like to exclude such solutions because, by causality, in such a situation the formation of

CTCs cannot be triggered by a human being. This motivates us to introduce the notion of *causal generation:* We say that a Cauchy horizon H is causally generated if there exists a partial Cauchy surface S and a compact set $Q \in S$ (the "factory" set) such that $H \subset J^+(Q)$. It seems that none of the present theorems preclude the creation of a causally but not compactly generated time machine without any singularity or violation of the WEC. A simple example of a causally but not compactly generated Cauchy horizon is the inner horizon of a Reissner-Nordstrom (or Kerr) black hole. It is trivial to show that in our solution the Cauchy horizon is causally generated. Intuitively, it seems that in Gott's spacetime the Cauchy horizon is not causally generated, but this is yet to be proved.

One might claim that it will be impossible for a human being to produce a Cauchy horizon if its generators are coming from some noncompact region. But the Reissner-Nordstrom example shows that this is not the case. For, at least in principle, one can manufacture a spherical charged object and then let it collapse. The (noncompactly generated) Cauchy horizon will then automatically form. There is, however, another argument against noncompact Cauchy horizons: If H is not compactly generated, then information from infinity could penetrate into the solution, making the evolution of CTCs to the future of H uncertain [6]. Nevertheless, this argument only shows that there is an uncertainty about the resultant geometry; it does not indicate that the formation of CTCs is impossible.

Moreover, there are situations in which, despite the above mentioned uncertainty, there are good reasons to expect that closed causal curves will indeed form. Consider a singularity-free spacetime in which CTCs evolve, from a noncompact initial slice S, to the future of a causally but not compactly generated Cauchy horizon H which itself includes an incomplete closed null geodesic. A simple example of such a spacetime can be obtained from the solution (1)-(6) by modifying the conformal factor F such that $t=t'$ becomes a boundary at infinity —e.g., by multiplying F by $(t-t')^{-2}$. (In this case H is no longer the past boundary of T, but it still can be shown that $N \subset H$.) Then, despite the ambiguity regarding the evolution beyond H, a causality violation is unavoidable: The evolution up to H is unambiguous, and H itself includes a closed causal curve. Can such solutions be constructed without any violation of the WEC? We do not have a clear answer, but nevertheless it appears that none of Tipler's and Hawking's theorems rule out this possibility.

To summarize, we have constructed a solution in which CTCs evolve from a reasonable initial slice S. In this solution the WEC is satisfied up to the moment where causality violation occurs. We then introduced the notion of causally generated Cauchy horizons, and proposed that, under certain conditions, the demand for compact generation should be relaxed if the Cauchy horizon is causally generated. We gave examples which show that (i) at least in principle, a human being has the ability to trigger the formation of a causally but not compactly generated Cauchy horizon, and (ii) although the future evolution beyond such a Cauchy horizon is in principle ambiguous, under certain conditions the evolution of closed causal curves (or even CTCs) is inevitable. It seems that the theorems of Tipler and Hawking do not rule out the existence of solutions of this kind which everywhere satisfy the WEC. We thus arrive at the following conclusion: *At present, one should not completely rule out the possibility of constructing a time machine from materials with positive energy densities.*

Of course, interpreting the above statement as if a time-machine construction may soon be in our hands would be a serious mistake. First, we recall that our success in overcoming the WEC problem is only partial; in the metric (1)-(6), we do expect a violation of the WEC somewhere at $t > t'$. [However, in the modified version described above, with the extra conformal factor $(t-t')^{-2}$, it is perhaps possible to completely avoid such a violation.] Second, there could be other problems which may prevent any attempt to violate causality. For example, it is not yet clear whether in our model one can avoid trapped or antitrapped surfaces and still satisfy the WEC. Perhaps the most serious problem is the instability of time-machine spacetimes. In particular, the focusing of future-directed null geodesics on N in our solution may indicate a classical instability (in addition to the quantum instability [6])—though this is yet to be verified. Yet, the author's point of view is that a better understanding of the backreaction effects is required before one completely rules out time-machine solutions due to their apparent instability.

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