Instability in Lattice Fracture

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The steady states of a crack moving in a triangular lattice are calculated and shown to become unstable at a certain velocity.

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There is a long-standing problem in the dynamics of fracture. Cracks in brittle materials are supposed to accelerate up to the Rayleigh wave speed [1,2] according to theory [3–5], while experiments seldom show them exceeding half this speed [6–9]. The problem is actually about energy balance. Cracks suddenly cost much more energy to propagate as they exceed a critical speed on the order of half the theoretical limit, but no one quite knows why.

There is no lack of possible mechanisms; as cracks pass the critical velocity the fracture surface becomes rough [7,8,10], the tip of the crack heats up by hundreds of degrees [11,12], and the crack is loud, emitting sound over a wide range of frequencies [13]. The difficulty lies in the fact that these phenomena have been hard to derive from the continuum theory that describes crack motion [14]. Cracks have had low terminal speeds in some numerical simulations [15,16], but the simulations have been as difficult to interpret as the experiments.

Slepyan [5,17] first discovered that the dynamic fracture of lattices [18] provides a natural setting in which to try to resolve this problem. In a series of beautiful calculations he analyzed cracks moving in simple two-dimensional lattices, and found that they naturally involve the emission of high-frequency waves. The main goal of this Letter is to show that steady-state cracks in a triangular lattice become unstable at a critical velocity, and begin to undergo oscillations with features reminiscent of experiment. One way to interpret this result is that it resolves ambiguities [3] in a continuum instability first noted by Yoffe [19]. The calculations do not yet constitute a quantitative explanation of fracture experiments, but from a qualitative point of view they are encouraging.

Figure 1 describes the model we will consider. A crack moves in a lattice strip composed of $2(N+1)$ rows of mass points. All of the bonds between lattice points are brittle-elastic, behaving as perfect linear springs until the instant they snap, from which point on they exert no force. The location of each mass point is described by a single spatial coordinate $u(m,n)$, which can be interpreted as the height of mass point (m,n) into or out of the page [20]. The index m takes integer values, while n takes value of the form $\frac{1}{2}, \frac{3}{2}, \ldots, N+\frac{1}{2}$. The model is described by the equation

$$\ddot{u}(m,n) = -b\dot{u} + \frac{1}{2} \sum_{\substack{\text{nearest} \\ \text{neighbors} \\ m',n'}} \mathcal{F}[u(m',n') - u(m,n)] , \tag{1}$$

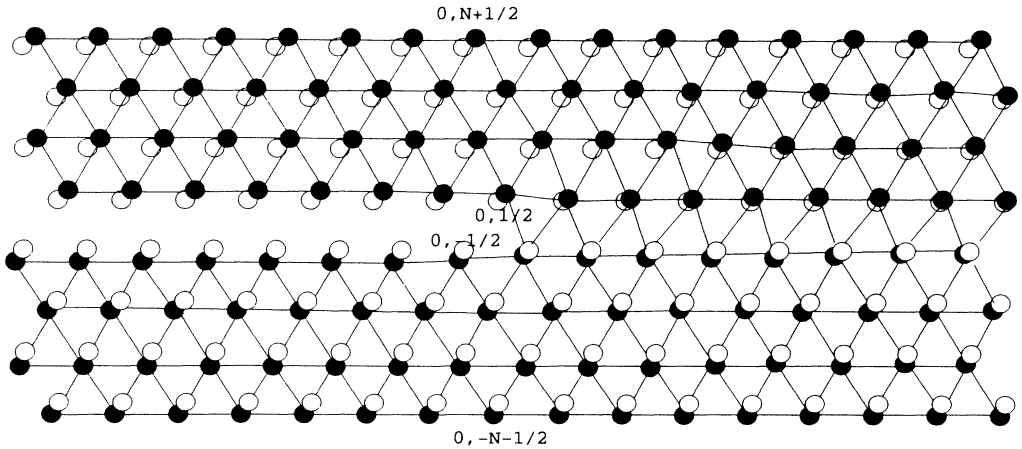

FIG. 1. Lattice model of fracture. The equilibrium locations of mass points are indicated by the white dots, while the black dots indicate the displacments $u(m,n)$ of mass points out of the page once stress is applied. The top line of dots is displaced out of the page by amount $u(m,N+\frac{1}{2}) = \Delta\sqrt{2N+1}$, and the bottom line into the page by amount $u(m,-N-\frac{1}{2}) = -\Delta\sqrt{2N+1}$. Lines connecting mass points indicate whether the displacement between them has exceeded the critical value of 2 [see Eq. (2)]. The crack tip has just reached location $m=0$.

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with

$$\mathcal{F}(u)=u\theta(2-u) \tag{2}$$

representing the brittle nature of the springs, θ the step function, and b the coefficient of a very small dissipative term. The boundary condition on the upper and lower surfaces of the strip is

$$u(m,\pm[N+\tfrac{1}{2}])=\pm\sqrt{2N+1}\Delta e^{-\alpha|t-m/v|}. \tag{3}$$

This boundary condition strains the lattice and provides the driving force for crack motion. The constant α is very small, and the form of the boundary condition indicates that one plans to search for a steady state moving at velocity v. The factor of $\sqrt{2N+1}$ simplifies subsequent results.

A steady state in a lattice is more complicated than one in a continuum; it is a configuration which repeats itself after a time interval a/v, but moved over by one lattice spacing. Taking the lattice spacing a to be 1, $u(m,n,t)=u(0,n,t-m/v)$, which means that all horizontal spatial behavior is contained in the time history of any single point on a horizontal line. We now suppose that a crack runs along the center of the strip. One by one, the bonds connecting $u(m,\frac{1}{2})$ with $u(m+1,-\frac{1}{2})$ or $u(m,-\frac{1}{2})$ break. They break because the distance between these points exceeds the limit set in Eq. (2) and as a consequence of the driving force described by Eq. (3). Supposing that these are the only bonds which snap (an assumption to which we will later return) it is easy to calculate the motion of all the mass points above the line $n=\frac{1}{2}$ as a function of the mass points on the line $n=\frac{1}{2}$, since in any region where the bonds do not snap the model has simple traveling wave solutions. Having thus solved for behavior in the vertical direction, and having exchanged behavior in the horizontal direction for time history, the whole problem reduces to an integral equation for the time evolution of a single mass point, $u(t)$. The most convenient variable is one which describes the bonds which break along the crack path:

$$u(t)=[u(0,\tfrac{1}{2},t)-u(0,-\tfrac{1}{2},t)]/2. \tag{4a}$$

In steady state, one has the symmetry

$$u(m,\tfrac{1}{2},t)=-u(m,-\tfrac{1}{2},t-1/2v), \tag{4b}$$

using which one obtains

$$u^{+}(\omega)Q(\omega)+u^{-}(\omega)=\frac{\Delta}{\sqrt{2N+1}}\frac{2\alpha}{\omega^2+\alpha^2}. \tag{5}$$

In this expression,

$$u^{\pm}(\omega)=\int_{-\infty}^{\infty}d\omega e^{i\omega t}u(t)\theta(\pm t), \tag{6}$$

$$Q(\omega)=\frac{F(\omega)}{F(\omega)-1-\cos(\omega/2v)}, \tag{7a}$$

$$F(\omega)=\left\{\frac{y^{[N-1]}-y^{-[N-1]}}{y^{N}-y^{-N}}-2z\right\}\cos(\omega/2v)+1, \tag{7b}$$

and

$$y=z+\sqrt{z^2-1}\ \text{ with }\ z=\frac{3-\cos(\omega/v)-\omega^2-ib\omega}{2\cos(\omega/2v)}. \tag{7c}$$

It is assumed in writing Eq. (5) that the bond between $u(0,\frac{1}{2})$ and $u(0,-\frac{1}{2})$ breaks at $t=0$. In order for that to be true, one must pose the extra condition

$$u(t=0)=1. \tag{8}$$

Recall that α is very small, so that the right-hand side of Eq. (5) is a delta function.

Equation (5) is of a type which can be solved by the Wiener-Hopf technique [22]. The trick is to write

$$Q(\omega)=Q^{-}/Q^{+}, \tag{9}$$

where Q^{-} is free of poles and zeros in the lower complex ω plane and Q^{+} is free of poles and zeros in the upper complex plane. Then Eq. (5) can be separated into the sum of two pieces, one of which has poles only in the lower half plane and one of which has poles only in the upper half plane, and being equal, must equal a constant. For $u(t)$ not to be singular at $t=0$ the constant must be zero, and one has

$$u^{-}(\omega)=\frac{\Delta}{\sqrt{2N+1}}\frac{Q^{-}(\omega)}{Q^{-}(0)(\alpha+i\omega)} \tag{10}$$

and a similar expression for u^{+}.

The analysis can proceed along two complementary paths from this point. On the one hand, one can solve Eq. (10) directly, using the fast Fourier transform. The decompositions required for Eq. (9) are carried out by taking the logarithm of Q, Fourier transforming, multiplying by an appropriate step function, inverting the transform, and exponentiating. This procedure is successful since $\ln Q(\omega)$ vanishes for large values of ω. However, one discovers that there are roots and poles of Q near the real axis, kept away from it only by small imaginary parts proportional to b. Physically, these roots and poles correspond to traveling waves induced by the crack tip. One way around this problem is to insist on resolution in the Fourier transforms that is much finer than the scale set by the dissipation b. Another possibility is to remove these roots before carrying out the transforms and to treat their product separately. This procedure has analytical advantages as well.

The second path for analysis of Eq. (10) notes that for large ω, $u^{-}\sim 1/(\alpha+i\omega)$. This large ω behavior describes a discontinuity in u at $t=0$, giving

$$u(t=0)=\frac{\Delta}{\sqrt{2N+1}}\frac{Q^{-}(\infty)}{Q^{-}(0)}. \tag{11}$$

This quantity is exactly what is needed to be found when bonds snap, as in Eq. (8). In order to proceed one divides the real (up to terms of order b) roots r_i and real (up to order b) poles p_i out of Q as follows:

$$\tilde{Q}(\omega)=\prod_i \frac{\omega-p_i^{\pm}}{\omega-r_i^{\pm}} Q(\omega). \quad (12)$$

The poles and roots are divided into two categories: Those with a superscript + are to be grouped with Q^+ and therefore have very small imaginary parts below the real axis, while those with superscript − go with Q^- and have very small imaginary parts above. $\tilde{Q}$ has a symmetry which does not hold for Q, namely, $\tilde{Q}(\omega)=\tilde{Q}(-\omega)$ (up to terms of order b). Using this symmetry, it is easy to show that

$$\tilde{Q}^-(\omega)\tilde{Q}^+(-\omega)=\tilde{Q}^-(0)\tilde{Q}^+(0). \quad (13)$$

One uses this relation, the regularity of $Q^-(\omega)$ at infinity, Eq. (12), Eq. (11), Eq. (8), and Eq. (9), to obtain

$$u(t=0)=1=\Delta\prod_i \left(\frac{r_i^+ p_i^-}{r_i^- p_i^+}\right)^{1/2}. \quad (14)$$

In order to evaluate this expression, one has to find the real roots and poles of Q. The number of these is proportional to the height of the strip N, but there is a numerical procedure that finds them extremely quickly. These results are somewhat different from those of Slepyan [17], because we have worked in a strip rather than an infinite plate [23], a choice that makes it possible to check numerical and analytical methods against one another.

Our results are summarized in Fig. 2, which was produced by calculating Δ from Eq. (14) for $N=100$, and many values of v. What the diagram illustrates is that for a given external driving force Δ many steady states are theoretically possible, each with different roots and poles $r_i p_i$, and therefore emitting different quantities of radiation. This radiation is at frequencies on the order of v, in units where the lattice spacing is 1, so in most realistic settings it would manifest itself experimentally as heat.

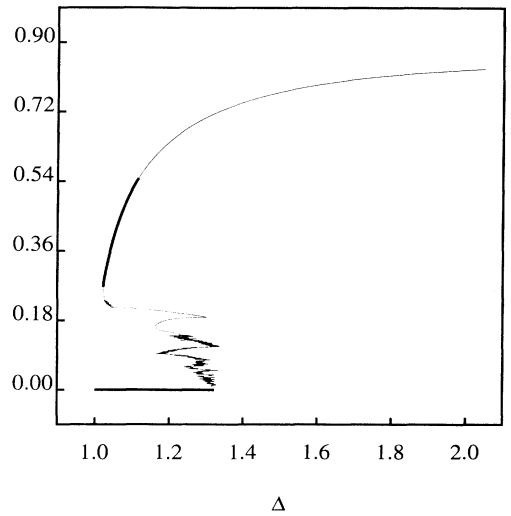

FIG. 2. Steady-state velocities as a function of external strain Δ. The velocity v is measured in units where the lattice spacing is 1, and the long-wavelength sound speed is $\sqrt{3}/2$. The thick lines indicate cases in which the steady states are known to be stable. Zero velocity states at strains $\Delta > 1$ correspond to the phenomenon of lattice trapping (Ref. [21]).

Finding steady states in this way is not sufficient to say that they are physically acceptable. Steady states with $0 < v < 0.244$ do not exist, because they violate the condition that $t=0$ must be the earliest time at which the bond between $u(0, \frac{1}{2})$ and $u(0, -\frac{1}{2})$ reaches 2. For v in this range, explicit calculation of the solutions shows that this bond stretches to a distance of more than 2 at negative times (without breaking), snaps back, and reaches 2 again at $t=0$. Similarly, the states with $v > v_c = 0.577$ do not exist. For $N=9$, at a velocity of $v_c = 0.577\ldots$, $\Delta_c = 1.158\ldots$, in units where the lattice spacing is 1 [19] the bond between $u(0, \frac{1}{2})$ and $u(1, \frac{1}{2})$ reaches a distance of 2 after the bond between $u(0, \frac{1}{2})$ and $u(0, -\frac{1}{2})$ snaps. The steady-state solutions strained with larger values of Δ are inconsistent; only dynamical solutions more complicated than steady states, involving the breaking of bonds off the crack path, are possible. To investigate these states, we return to Eq. (1) and numerically solve the model directly.

Some results from these direct numerical investigations are contained in Fig. 3. The diagram shows patterns of broken bonds left behind the crack tip. Just above the threshold at which horizontal bonds begin to break, one

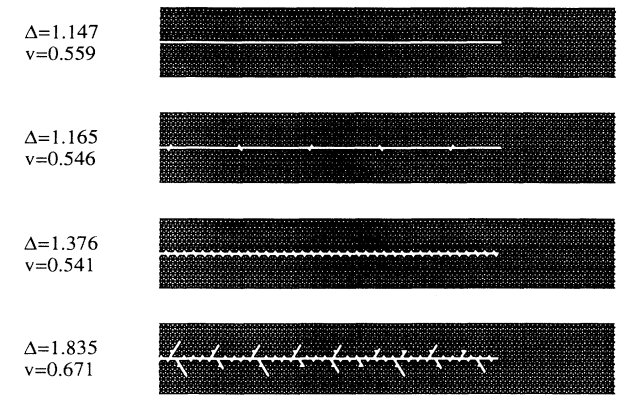

FIG. 3. Pictures of broken bonds left behind the crack tip at four different values of Δ. The top figure shows the simple pattern of bonds broken by a steady-state crack. At a value of Δ slightly above the critical one where horizontal bonds occasionally snap, the pattern is periodic. Notice that the mean velocity decreases relative to the steady state, although the external strain has increased. As the strain Δ increases further, other periodic states can be found, and finally states with complicated spatial structure.

expects the distance between these extra broken bonds to diverge. The reason is that breaking a horizontal bond takes energy from the crack and slows it below the critical value. The crack then tries once more to reach the steady state, and only in the last stages of the approach does another horizontal bond snap, beginning the process again [24]. Analysis of linear stability shows that convergence to steady states is exponential, and occurs exactly at rate b, where b is the damping in Eq. (1). This picture leads to the conclusion that the frequency v with which horizontal bonds snap should scale above the critical strain Δ_c as

$$v \sim -b/\ln(1-\Delta_c/\Delta)\,, \tag{15}$$

a result that is consistent with our numerics, but hard to check conclusively. Further increasing the external strain Δ makes a wide variety of behavior possible, including quasiperiodicity and low-dimensional chaos.

A similar instability was predicted long ago in the framework of continuum theory by Yoffe [19], but its consequences, and the precise velocity at which it should appear, have always been unclear [3]. These uncertainties are resolved by the lattice model. However, the lattice model does not immediately explain all the experimental puzzles. Experiments [8] see crack tip oscillations at a definite—very low—frequency, which sets in at around 40% of the relevant sound speed. The model provides a mechanism for producing low frequencies [Eq. (15)], but not as low as seen experimentally, and does not produce them at velocities less than 66% of the sound speed [25]. On the other hand, the bottom panel of Fig. 3 resembles actual experimental fracture surfaces [26], and too many extensions to the model deserve examination for it to be dismissed at this stage. What happens when the mass points move with 2 degrees of freedom rather than 1 as here? What are the effects of thermal noise? What are the effects of random bond strengths? What would happen in a random lattice? Do the bonds broken off axis lead to microcracks or to dislocations? These are some of the points that deserve further investigation.

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[26] Examination of the fracture surface under a microscope shows many cracks branching off the main crack path.