Universal Anharmonic Vibrator Description of Nuclei and Critical Nuclear Phase Transitions

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A new analysis of yrast energies in collective medium and heavy even-even nuclei shows two startling results. First, although these nuclei exhibit widely varying structure [the ratio $E(4_1^+)/E(2_1^+)$ ranges from ~2.0 to ~3.33], they show a universal behavior of an *anharmonic vibrator* with *nearly constant anharmonicity* in which the multiphonon interactions are independent of the internal structure of the vibration. Second, the transition from anharmonic vibrator to rotor can be described by *critical phase transitional behavior.* These results suggest a reexamination of very basic ideas of structural evolution in finite-body nuclear systems.

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Although the lowest levels of even-even nuclei are well-known signatures of structure, there has been little study of their energy relationships over the entire nuclear chart. Of course, it is well known that $E(2_1^+)$ and $E(4_1^+)$ decrease through the course of a vibrator → rotor shape transition. Also, the ratio $R_{4/2} = E(4_1^+)/E(2_1^+)$ increases from ~1.2–1.6 near doubly magic nuclei to ~2.0–2.2 in vibrational nuclei, 2.5–3.0 in transitional species, and ~3.33 in well-deformed symmetric rotor nuclei. Surprisingly, however, the global relationships of $E(4_1^+)$ to $E(2_1^+)$ have never been probed nor have their remarkable implications been interpreted.

It is the purpose of this Letter to do so and to show that rather startling conclusions result which may alter our understanding of the structure of most nuclei, of how nuclei pass through transitional regions, and of the universality of the observed phenomenology.

Figure 1 presents three plots of $E(4_1^+)$ against $E(2_1^+)$. The plots include *all* even-even nuclei with $Z \geq 38$ with collective $R_{4/2}$ values between the near-harmonic value 2.05 and the near-rotor value of 3.15. No nuclei, magic or otherwise, are excluded. There are three striking and heretofore unrecognized features of Fig. 1. (i) In each plot, the data are highly correlated along a straight line; (ii) these lines are *parallel* to the vibrator limit, with a *slope* of ~2.0; and (iii) the plots for each region have very similar offsets from the vibrator limit. Points (i) and (ii) suggest that these data satisfy the simple formula

$$E(4_1^+) = \alpha E(2_1^+) + \varepsilon_4 , \qquad (1)$$

in which $\alpha \sim 2.0$. Linear least-squares fits give the following: for $Z=38$–50, $\alpha = 1.96 \pm 0.04$, $\varepsilon_4 = 195 \pm 24$ keV; for $Z=52$–82, $\alpha = 2.00 \pm 0.04$, $\varepsilon_4 = 175 \pm 24$ keV for $N=52$–82 and $\alpha = 2.00 \pm 0.04$, $\varepsilon_4 = 148 \pm 14$ keV for $N=84$–126; for $Z > 82$, $\alpha = 2.08 \pm 0.05$, $\varepsilon_4 = 67 \pm 16$ keV.

Point (iii) means that ε_4 is nearly unchanged from region to region (except for the actinides). This is vividly highlighted by combining the data for all nuclei from $Z=38$ to 82 into a *single plot* as in Fig. 2. The data still are highly correlated, along a line parallel to the harmonic vibrator line, despite the fact that they cover such a wide mass range and include vibrational regions as dispersed Zr, Cd, and Hg, and transitional nuclei in regions as different as the near $N=Z$ region around $A=80$, the extremely rapid $A=100$ region near $Z=38$, the gradual $A \sim 130$ region, the $A=150$ vibrator → rotor region, and the γ-soft nuclei near $A=130$ and 190. An unexpected mass *independence* of $E(2_1^+)$ is implicit: For example, ^{102}Mo, ^{148}Nd, and ^{190}Pt all have the same $E(2_1^+)$ values (299 ± 3 keV) and the same $R_{4/2}$ values

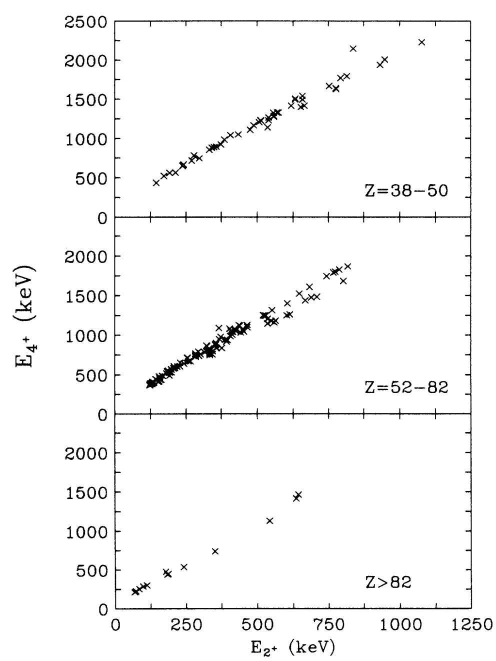

FIG. 1. Plots of $E(4_1^+)$ and $E(2_1^+)$ for three mass regions for the nuclei with $R_{4/2}$ values between 2.05 and 3.15. Data are from Ref. [1].

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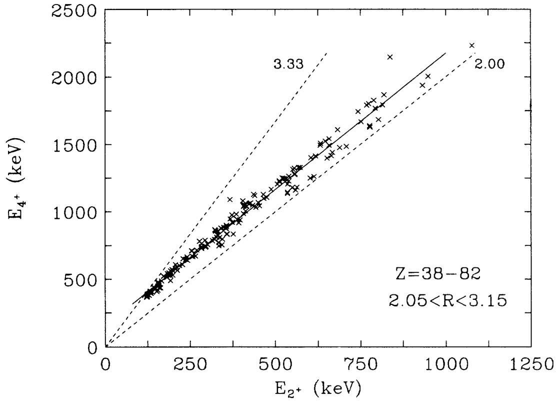

FIG. 2. Combination of data, such as shown in Fig. 1, for nuclei from $Z=38-82$, inclusive. Extension of the plot to lower or higher Z does not materially alter the results. The harmonic vibrator ($R_{4/2}=2.00$) and rotor ($R_{4/2}=3.33$) limits are shown as well as a linear least-squares fit to the data.

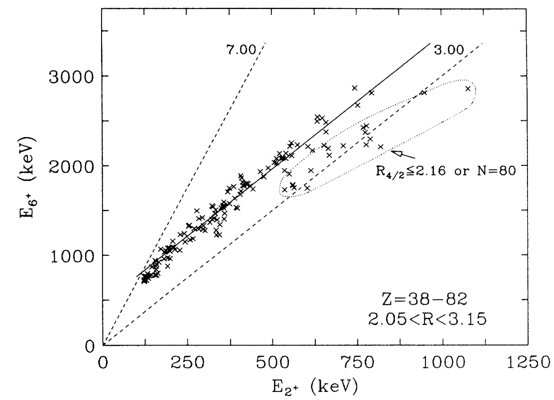

FIG. 3. Similar to Fig. 2 except for $E(6_1^+)$. The harmonic vibrator ($R_{6/2}=3.00$) and the rotor ($R_{6/2}=7.00$) limits are shown as well as the predicted function $E(6_1^+)=3.00E(2_1^+)+3\varepsilon_4$, where $\varepsilon_4=156$ keV. The elliptically encircled points are anomalous values off the main trend, which have either $R_{4/2}\lesssim 2.16$ or $N=80$.

(2.50 ± 0.01). Apparently, it is only in rotational nuclei that yrast energies in medium and heavy mass nuclei approach an $A^{-5/3}$ dependence. A linear least-squares fit of Eq. (1) to Fig. 2 gives $\alpha=2.02\pm0.02$ and $\varepsilon_4=156\pm10$ keV: That is, within errors, all the nuclei, from $Z=38-82$, with $2.05\leq R_{4/2}\leq 3.15$, can be described by the expression

$$E(4_1^+)=2E(2_1^+)+\varepsilon_4\,, \tag{2}$$

which is that of an anharmonic vibrator where $\varepsilon_4=156$ keV is the deviation of the two-phonon 4_1^+ energy from the harmonic value. Although we stress the average behavior of the overall trend in Fig. 2, we also note that each individual isotopic or isotonic series also follows a linear path with slope close to 2.0. The only notable deviations in Fig. 2 from Eq. (2) are two nuclei, ^{94}Sr and ^{184}Hg, lying well above the line, a small group of $N=88$ nuclei slightly below the line at $E(2_1^+)\sim350$ keV, and a few nuclei with high 2_1^+ energies, most of which have two valence nucleons of one type. Interestingly, though, the overall scatter is statistical: The 1σ deviation from the line is 5% and the moments of the distribution are consistent with a Gaussian distribution. We conclude that nearly all these nonrotational nuclei can be described by a *single anharmonic vibrator equation with constant anharmonicity* ε_4. We pursue below the meaning and implications of this startling result.

This anharmonic vibrator interpretation seems so bizarre that one wonders if its success in describing the systematic trend for so many nuclei with constant parameters might be accidental. This question can be tested by asking if other yrast levels can likewise be explained. Fortunately, this is easy since there are relations [2] between the anharmonicities of two-phonon and higher-phonon vibrational states. The three-phonon 6^+ energy depends only on $E(4_1^+)$ since it is made by coupling a one-phonon 2^+ excitation with the 4^+ two-phonon state. Since the three phonons are indistinguishable, we have $E(6_1^+)=3E(2_1^+)+3\varepsilon_4$. We test this in Fig. 3. The trend of the 6^+ data is linear in $E(2_1^+)$, in excellent agreement with the above prediction using the *same* ε_4 as in Eq. (2). The only exceptions are a few 6^+ levels in nuclei with large $E(2_1^+)$ that either have $N=80$ or near-harmonic vibrator $R_{4/2}$ values. These anomalous levels deserve further study but do not impact the main point that an anharmonic vibrator with constant anharmonicity describes the low-spin yrast levels of nearly all nonrotational medium and heavy nuclei. The 8_1^+ energies for the same nuclei show somewhat larger fluctuations but are in good agreement with the anharmonic vibrator prediction $E(8_1^+)=4E(2_1^+)+6\varepsilon_4$.

We next note an important aspect of Eq. (2). While the slope is constant, that is, $\Delta E(4_1^+)=2.0\Delta E(2_1^+)$, the *ratio* $R_{4/2}\equiv E(4_1^+)/E(2_1^+)$ varies from ~2 to nearly 3.33, implying that the underlying structure is changing. This compatibility of constant slope and constant anharmonicity with changing $R_{4/2}$ results from the interplay of the two terms in Eq. (2), such that as $E(2_1^+)$ drops, the ε_4 contribution to $E(4_1^+)$ becomes relatively larger, raising the $4_1^+/2_1^+$ energy ratio, as seen by rewriting Eq. (2) as

$$R_{4/2}=2.0+156/E(2_1^+)\,. \tag{3}$$

The apparent dichotomy between changing nuclear structure but a universal anharmonic vibrator description with constant anharmonicity actually points toward a deep underlying implication. To see this we note that the energy relations for yrast levels just discussed are particular cases of a very general result for any anharmonic vibrator with up to two-body interactions, namely,

$$E(J)=nE(2_1^+)+\frac{n(n-1)}{2}\varepsilon_4\,, \tag{4}$$

where $n=J/2$ is the phonon number. This result is even more general than the specific form of Eq. (4) indicates: Equation (4) gives the energy of the aligned coupling of any n-phonon state $E(n)$, if $E(2_1^+)$ is replaced by the one-phonon energy and ε_4 is the anharmonicity of the two-phonon level. Equation (4) holds independent of the internal structure of the phonon. Thus, Eq. (4) describes, for example, the yrast energies of γ-soft nuclei [Wilets-Jean or $O(6)$ model] ($R_{4/2}=2.50$) or even of a symmetric rotor ($R_{4/2}=3.33$), and it can be written in the form [3] $E(J)=aJ+bJ(J-2)$ or that of the Ejiri relation [4] $E(J)=aJ+bJ(J+1)$. The remarkable aspect of Eq. (4) is therefore not at all its success in individual nuclei—it describes a variety of structures—but rather the completely unexpected constancy of ε_4 over such a vast span of nuclei comprising a wide variety of mean field structures. This constancy implies that while the internal quadrupole phonon structure (the 2_1^+ state) may be that appropriate to a spherical vibrator or a γ-soft nucleus, or to some transitional type or even a near rotor, nevertheless, independent of this phonon structure, the anharmonicity—that is, the two-body phonon-phonon interaction—somehow remains constant. In the complementary Ejiri formulation, the constancy of ε_4 corresponds to a (also heretofore unrecognized) constancy of the coefficient b and hence to a constancy of the rotational perturbation to the changing vibrational energy.

A final point is that these results imply much more than the linearity of the ratio $R_{J/2}$ with $R_{4/2}$ observed in Mallmann plots [5]. A simple manipulation of Eq. (4) shows the ratio plot effectively *eliminates* ε_4 and, therefore, a linear Mallmann plot automatically results from *any* two-parameter energy-angular momentum relation. Moreover, even the specific linear trend seen in the empirical Mallmann plot of $R_{6/2}$, namely $R_{6/2}\sim 3R_{4/2}-3$, only implies that $E(J)$ is proportional to J and/or J^2: It says nothing further, nor does it imply a constancy of ε_4 (or of b in the Ejiri relation).

We now proceed further by considering the extension of the phenomenology of Figs. 1 and 2 into the rotational region ($R_{4/2}>3.15$) which will lead to additional unexpected and remarkable results. Clearly, Eq. (2) cannot continue to apply for rotational nuclei since, as $E(2_1^+)$ drops, at some point $E(2_1^+)$ becomes less than $\varepsilon_4/1.33$ and then Eq. (2) give $E(4_1^+)>3.33E(2_1^+)$ which would cross the rotor limit. This is illustrated in the inset to Fig. 4, which focuses on the $Z=50$–82, $N=82$–126 region (other regions behave similarly). As the data approach the rotor limit, within a very narrow range of $E(2_1^+)$ values the trend curves downward away from the extension of Eq. (2), asymptotically merging into the rotor line labeled 3.33. Equation (2), however, can still describe the data provided it is modified to include an additive term which is a *function* of $E(2_1^+)$. We thus rewrite

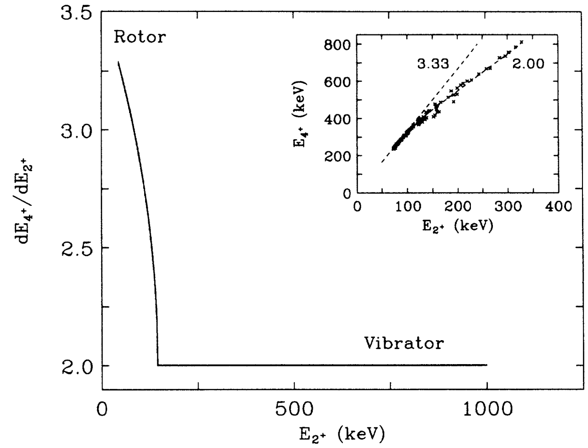

FIG. 4. Inset: data for nuclei with $E(2_1^+)<330$ keV isolating the behavior of near-rotational nuclei. The curve is a least-squares fit. Main plot: the derivative $dE(4_1^+)/dE(2_1^+)$ against $E(2_1^+)$ obtained from the fits in Figs. 2 and 4 (inset) with Eqs. (2) and (6). Note that other functional forms for the last term in Eq. (5) also give very sharp increases in the region $E(2_1^+)\sim\varepsilon_4$.

Eq. (2) as

$$E(4_1^+)=2.0E(2_1^+)+\varepsilon_4\{1-a[E_c(2_1^+)-E(2_1^+)]^{\lambda+1}\}\quad [E(2_1^+)<E_c(2_1^+)]\,, \tag{5}$$

where $E_c(2_1^+)$ marks the start of the transition region. The last term introduces an $E(2_1^+)$ dependence to the anharmonicity ε_4. Taking $\varepsilon_4=148$ keV (for this mass region), a least-squares fit gives $a=0.0007$ keV$^{-(\lambda+1)}$, $\lambda=0.49$, and $E_c(2_1^+)=145$ keV.

Since Fig. 4 (inset)—and Eq. (5)—exhibit a changing slope of $E(4_1^+)$ against $E(2_1^+)$, it is interesting to explicitly consider the derivative $dE(4_1^+)/dE(2_1^+)$ which has limiting values of 2.0 for the vibrator and 3.33 for the rotor. Note that these derivatives are *not* the same as $R_{4/2}$. Decades of experience with nuclear structure would suggest that $dE(4_1^+)/dE(2_1^+)$ exhibits a curved and gradual trajectory against $E(2_1^+)$ between the vibrator and rotor fixed points. In contrast, the actual result, from Eq. (5), is

$$dE(4_1^+)/dE(2_1^+)=2+C[E_c(2_1^+)-E(2_1^+)]^{\lambda}\,, \tag{6}$$

where $C=\varepsilon_4 a(\lambda+1)=0.15$ keV$^{-\lambda}$ for $E(2_1^+)<E_c(2_1^+)$ (and 0 otherwise), which reveals the totally different behavior shown in the figure. Of course, the fit to the data perforce reflects only the average behavior: Strings of particular nuclides (e.g., isotopes and isotones) may well have somewhat diverse, fluctuating, irregular behavior. Nevertheless, the curve shows that, on average, the empirical derivative $dE(4_1^+)/dE(2_1^+)$ cuts *horizontally* across the plot for nearly all 2_1^+ energies, and then suddenly turns nearly vertically upward towards 3.33 in a *very narrow range* of $E(2_1^+)$ energies ($\sim$130–145 keV).

This is a direct consequence of the constancy of ε_4 in Eq. (2) for most nuclei and the deviation from Eq. (2) given by Eq. (5) for near-rotor nuclei.

The surprisingly sharp kink in Fig. 4 reveals a behavior typical of true phase transitions in many-body systems [6]. Indeed, Eq. (6) has exactly the form of a critical phase transition characterized by a critical point, by an order parameter that is zero above the critical point and finite below it, and by a power law with a critical exponent λ. [An example of an order parameter is the net magnetization of a ferromagnet above and below the critical temperature (Curie point).] Here, $E_c(2_1^+)$ is the critical point, and the nuclear order parameter would be

$$O(4/2) \equiv dE(4_1^+)/dE(2_1^+) - 2\,. \tag{7}$$

The intriguing result found in all mass regions, *that the offset in the anharmonic vibrator region is related to the critical point,* namely, that $E_c(2_1^+) \sim \varepsilon_4$, is natural in view of the requirement deduced earlier that rotational behavior must set in for $E(2_1^+) \sim \varepsilon_4/1.33$, and it gives an interesting general rule that is familiar in a qualitative way but which here is obtained specifically from the fits. It follows from Eq. (2) and the result $E_c(2_1^+) = \varepsilon_4$ that, at the critical point,

$$E_c(4_1^+) = 2E_c(2_1^+) + E_c(2_1^+) = 3E_c(2_1^+)\,. \tag{8}$$

The value $R_{4/2} = 3.0$ thus takes on special significance as universally marking the start of the phase transition which ensues for nuclei with $3.00 < R_{4/2} < 3.33$.

Normally, transitional regions have been associated with a shift from a potential centered at $\beta = 0$ to one with lower symmetry centered at finite β. Our analysis suggests a reexamination of phase transitional behavior as the average and rms deformations associated with zero point motion change as the nuclear potential evolves and it raises the question of how critical phase transitional behavior can develop in the (valence nucleon dominated) few-body nuclear environment.

Finally, since these 2_1^+ and 4_1^+ data have long been known, one might ask why our approach yields such a new interpretation. The reason is simply that, first, we plot against $E(2_1^+)$ rather than, say, N, Z, or A and, second, we plot $E(4_1^+)$ instead of $R_{4/2}$ against $E(2_1^+)$. Because of the constant in Eqs. (1) and (2), $R_{4/2}$ changes smoothly against $E(2_1^+)$, obscuring the linearity in the anharmonic region and masking the phase transitional behavior. Similarly, double ratio plots of the Mallmann type cannot yield a unique relation between energy and angular momentum: In fact, they automatically eliminate any information on ε_4 and therefore are insensitive to the phonon anharmonicity. However, when $E(4_1^+)$ is directly plotted against $E(2_1^+)$, the constant slope reveals the constancy of vibrational anharmonicity, along with the sudden change occurring in the phase transitional region.

To summarize, for all collective nuclei from $Z = 38$ to 82 (and nearly as well from ^{12}C to the actinides), 4_1^+ energies follow a *universal* trend against $E(2_1^+)$. The data lie on the *same* straight line with slope 2.0. These nuclei, of *widely varying* structure ($2.05 \le R_{4/2} \le 3.15$), can be described by a generic *anharmonic vibrator* equation with *nearly constant anharmonicity, independent* of internal phonon structure and nearly independent of mass region: $E(4_1^+) = 2.0E(2_1^+) + \varepsilon_4$ with $\varepsilon_4 \sim 150$ keV ($\varepsilon_4 \sim 70$ keV in the actinides). Second, the derivative, $dE(4_1^+)/dE(2_1^+) \sim 2.0$ until the data, in effect, hit the rotor limit: it then rapidly increases to 3.33. Its variations can be described by critical phase transitional behavior [see Eq. (6)]. In a given region, the phase transition ensues at $E(2_1^+) \sim \varepsilon_4$ (or $R_{4/2} \sim 3.0$) and rotational behavior sets in at $E(2_1^+) \sim \varepsilon_4/1.33$: These may be useful for predictions in new regions. These results suggest the need to reexamine the nature and evolution of nuclear structure, the universality of phonon and rotation-vibration interactions, and the nature of phase transitional behavior in finite-body nuclear systems.

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