Models of Fractal Cracks

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We present theoretical ideas which allow us to understand part of the scaling laws recently observed on branched cracks. We argue that some features are common to all critical branched structures, which barely survive when propagating. These ideas are illustrated by the directed percolation problem, which serves as an excellent toy model. Finally, we propose a Langevin equation for unbranched cracks in three dimensions, which naturally leads to self-affine structures.

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There has been a recent upsurge of activity in the statistical description of fracture surfaces. The motivation was triggered by the interesting observation [1] that these surfaces are scale invariant, critical objects in the sense that the local height z scales as the in-plane displacement $\mathbf{r}$ as $z \propto r^{\zeta}$, where $\zeta < 1$ is the roughness exponent: These surfaces are called self-affine fractals [2]—planar surfaces would correspond to $\zeta = 1$. Recent experiments spurred further interest: It is not only the "fractal" region which is surprisingly large (over three decades in some cases: 0.5 μm to 0.5 mm [3]), but also the roughness exponent ζ which seems to be quite independent of the material and fracture mode [4]. Experiments on aluminum alloys, bakelite, steels, concrete, etc., using very different techniques, all yield $\zeta \simeq 0.8$ [1,3,5–7]. This "universal" behavior is also found in numerical simulations of random fuse models [8,9], leading to roughness exponents independent of the microscopic disorder. These results strongly suggest that the model describing crack formation and propagation must exhibit scaling properties at large length scales, much as is found in nonlinear diffusion equations describing, e.g., growing surfaces [10] or flame fronts [11]. We shall indeed propose, in the last part of this paper, a "Langevin" equation for the propagation of unbranched cracks in three dimensions, which leads to this scaling behavior.

One must, however, appreciate that there are at least two rather different types of cracks. The first one is observed in the case of slowly propagating cracks. Although "rough," the fracture surface is univalued: Its locus can be described as a function $z(\mathbf{r})$. All the above cited experiments deal with this first type. The second one is found when the crack exceeds a certain critical velocity: As is well known [12,13], crack branching then occurs, resulting in a complicated fracture surface with many overhangs and secondary cracks. This motivated a new series of experiments [14], on polycrystalline Ni_3Al, the aim of which was to study the statistics of these branched cracks; see Fig. 1(a). A certain number of statistical properties were analyzed, in particular the fluctuations of the number of points $N(\mathbf{r})$ which have the same abscissa $\mathbf{r}$, or the probability $P(z,\mathbf{r})$ that the point $(z,\mathbf{r})$ belongs to the structure, knowing that (0,0) does. One finds that [14] (i) $N(\mathbf{r})$ typically grows as r^b with $b = 0.13 \pm 0.03$, and (ii) $P(z,\mathbf{r})$ takes a *scaling form*

$$P(z,\mathbf{r}) = r^{-\zeta} \mathcal{F}(z/r^{\zeta}) \tag{1}$$

with again $\zeta = 0.82 \pm 0.05$. Furthermore, the scaling function is extremely broad [see Fig. 2(a)]: $\mathcal{F}(u) \simeq u^{-1}$

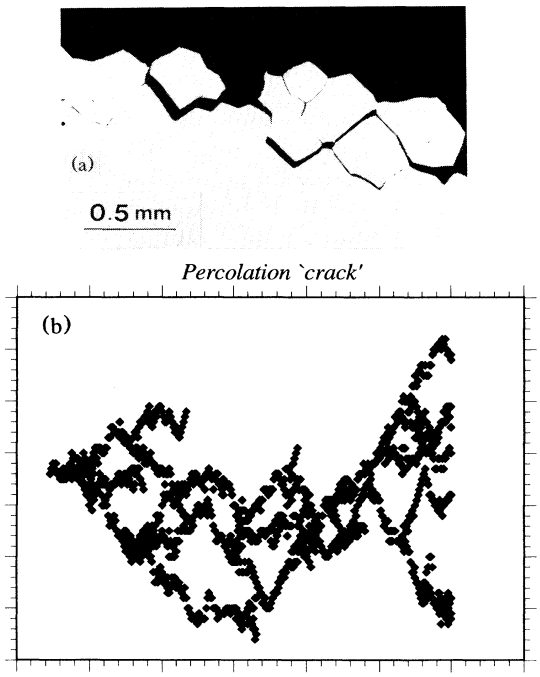

FIG. 1. (a) Typical 2D cut of branched crack observed on Ni_3Al. See Ref. [14] for more details. (b) Crack generated by numerically evolving a population of walkers obeying Eq. (2), with $A = A_c$.

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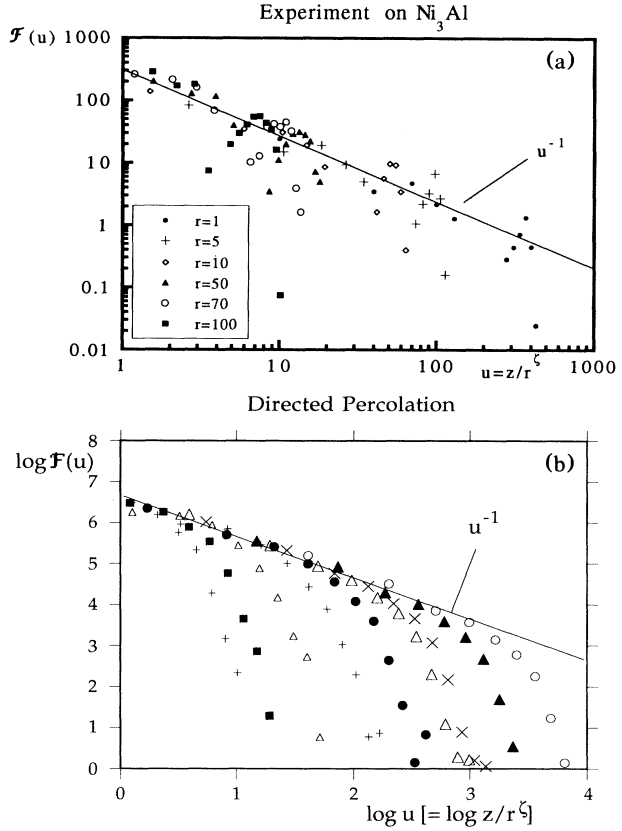

FIG. 2. (a) Plot of $\mathcal{F}(u)$ [defined by Eq. (1)], for different values of r, for the experimental cracks of Ref. [14]. (b) Same as (a), but for the directed percolation cluster problem, and for $r=1,2,4,8,\ldots$. Note that $P(r,z)$ is defined by averaging over all possible pair of points belonging to the structure. The departure from a u^{-1} behavior occurs earlier for large r since for a given picture of length L, $u_{\max}\equiv z_{\max}/r^{\zeta}\simeq(L/r)^{\zeta}$. We have used $L=500$.

for $1\leq u\leq z_{\max}/r^{\zeta}$. Hence $P(z,\mathbf{r})\simeq z^{-1}$ independently of r as soon as $z_{\max}\geq z\geq r^{\zeta}$. The crack can thus be pictured as a "fluffy fractal": Its self-affine backbone (with roughness exponent $\zeta\simeq 0.8$) is decorated by a density of branches decaying as z^{-1}, which extend up to $z_{\max}=z_0+Cr^{\zeta}$ ($z_{\max}$ is the total width of the structure, including branches; see [14]).

The aim of this Letter is to discuss these results from a theoretical point of view. We first suggest that two-dimensional directed percolation (DP) is an excellent toy model, which contains the important ingredients to reproduce the main properties reported above (although the value of the exponents are incorrect). We then show how these properties are common to all "critical" branched structures, i.e., to structures which barely survive without exponentially growing. We argue that fracture is *by essence* critical in that sense.

Let us thus start by analyzing a minimal model of crack branching in two dimensions. We assume that an existing crack may either propagate as a random walk, die, or give birth to a secondary crack. Furthermore, when two cracks come close to each other, one of them dies. The coarse-grained equation governing the time evolution of the crack density $\rho(z,r)$ may thus be written as

$$\frac{\partial\rho(z,r)}{\partial r}=D\frac{\partial^2\rho(z,r)}{\partial z^2}+A\rho(z,r)-B\rho^2(z,r)\,, \qquad (2)$$

where D is the "diffusion constant" of the cracks, A measures the difference between the branching probability and the death probability, and B models the competition between cracks. It is reasonable to assume that as the external stress is increased, A increases while B and D are roughly constant. Equation (2) has been thoroughly discussed in the literature [15–17]: For A less than a certain value A_c, the density ρ vanishes almost surely for large times: The crack does not propagate. For $A>A_c$, on the other hand, the density reaches a finite value. $A=A_c$ is thus a second order phase transition point, which is in the universality class of directed percolation. A typical crack obtained for $A=A_c$ is shown in Fig. 1(b). The critical exponents for this case are well known [15–18]: One finds in particular that the number of branches $N(r)$ grows as $r^{b_{\mathrm{DP}}}$ with $b_{\mathrm{DP}}=0.31$, and that the "width" of the structure grows as $r^{\zeta_{\mathrm{DP}}}$ with $\zeta_{\mathrm{DP}}=0.63$ (note the relation between b_{DP}, ζ_{DP}, and their usual notation: $b_{\mathrm{DP}}=\beta/\nu_{\parallel}$ and $\zeta_{\mathrm{DP}}=\nu_{\perp}/\nu_{\parallel}$). We have furthermore computed numerically the full distribution $P(z,r)$, *averaged over all points belonging to the structure* (as was done in [14]), and found that it is precisely given by Eq. (1), with $\zeta=\zeta_{\mathrm{DP}}$, and $\mathcal{F}(u)\simeq u^{-1}$ for large u; see Fig. 2(b).

This model is of course too simple to describe the three-dimensional problem of crack propagation. It shows, however, very clearly how, by increasing the external load, one progressively reaches the critical point where the crack just manages to "percolate" through the sample: This illustrates why scaling laws appear quite naturally in fracture. It is, however, conceivable that one may reach the regime $A>A_c$ by suddenly applying a sufficiently large stress, or at the end of the fracture process. In this case, one expects the self-similar regime to be of finite extent: $r<\xi\propto(A-A_c)^{-\nu}$, with $\nu=1.734$ [18]. Before turning to more general statements, let us add one remark. One may argue that due to the inherent microscopic disorder, A should be randomly position dependent: $A(z,r)$. One can, however, show, using a Harris-like argument, that since $\nu(1+\zeta_{\mathrm{DP}})=2.84>2$, disorder is "irrelevant." We have confirmed this by numerical simulations. If, however, B is zero, then Eq. (2) with a random $A(z,r)$ would be the equation for the partition function of directed polymers in a random environment (i.e., an elastic string interacting with random impurities) [19], which was already proposed in the context of *unbranched* cracks in [9,20,21]. The idea of Refs.

[9,21] is that a crack might minimize globally the fracture "energy" along its path, and hence would be identical to the ground state of a directed polymer in a random environment. An explicit model exhibiting this property was discussed in [9]. This leads to $\zeta_{2D}=\frac{2}{3}$ (rather close to the DP value and to 2D experiments [22]) but $\zeta_{3D}\simeq 0.4$–0.5 [23,24], far below the experimental value $\zeta=0.8$ (see, however, [4]). This whole argument assumes that the equation of motion of the propagating crack can be obtained by minimizing a certain (random) Hamiltonian. Dynamical equations possessing this variational property are, however, the exception rather than the rule, and the nonlinear Langevin equation [Eq. (4)] presented below is a counterexample. On the other hand, the minimal energy path *can be found dynamically* in the presence of branching: Branching is indeed a well known method to sample phase space thermodynamically (and is commonly used in Monte Carlo simulations [25]).

We now give a more general description of critical branched structures. We will define the probability per unit length that the backbone gives birth to a new offspring as l/l_1. The probability that this new structure survives at least for a length $r-r'$ will be denoted $K(r-r')$, in which case the number of branches it contains is $N(r-r')$. If the whole structure is self-similar, then the total number of branches after a length r is given by

$$N(r)=\int_0^r dr'\frac{1}{l_1}K(r-r')N(r-r')\,, \tag{3}$$

which leads to the following differential equation for $N(r)$: $dN(r)/dr=K(r)N(r)$. Now, if $K(r)$ decays faster than r^{-1}, $N(r)$ tends a constant for large r. If $K(r)$ decays as $r^{-\alpha}$ with $\alpha<1$, then $N(r)$ grows very fast with r: $N(r)\simeq\exp(r^{1-\alpha})$. $\alpha=1$ hence appears as a critical value where the branched structure barely survives. Writing $K(r)=l_2/r$, one finds $N(r)\simeq r^b$ with $b\equiv l_2/l_1$. From the definition of $K(r)$, $dK(r)/dr$ is the probability that the structure has a length equal to r. The average length $\bar{l}$ of the structure is thus given by $\bar{l}=\int_0^\infty dr\times r dK(r)/dr\equiv\int_0^\infty dr\,K(r)$. The critical point $\alpha=1$ is thus the point where $\bar{l}$ just diverges. This r^{-1} behavior of $K(r)$ is found analytically in the well known problem of branching on a Bethe lattice (for which $b=2$) [26], and also, very recently, on various tree structures studied numerically [27,28]. A general argument for such a behavior can thus be constructed, which furthermore shows that the exponent b measures the ratio of two important length scales: the typical length l_2 of one branch and the distance l_1 between branches. b was found to be $\simeq 0.13$ in the experiments on Ni_3Al, which is compatible with the visual aspect of the cracks.

Now, defining $N(z,r)$ the number of points belonging to the structure with coordinates z,r, one has, from the definition of $P(z,r)$, $N(z,r)\equiv N(r)P(z,r)$. One can write a general equation obeyed by $N(z,r)$ as

$$N(z,r)=\int_0^r dr'(1/l_1)K(r-r')\left[\int_{-\infty}^{+\infty}dz'P(z',r')N(z-z',r-r')\right]. \tag{4}$$

Now, assuming the scaling form given in Eq. (1) and $K(r)=l_2/r$, we find that the function $\mathcal{F}$ has to satisfy the following relation:

$$\mathcal{F}(u)=b\int_0^1 dv\int_{-\infty}^{+\infty}dw(1-v)^{b-1}v^{-\zeta}\mathcal{F}\left[\frac{u}{v^\zeta}-w\frac{(1-v)^\zeta}{v^\zeta}\right]\mathcal{F}(w)\,. \tag{5}$$

If one looks for a solution of Eq. (5) of the form $\mathcal{F}(u)\sim u_0^\mu/u^{1+\mu}$ for large u, then μ is found to obey the equation $b\Gamma(b)\Gamma(\zeta\mu+1)=\Gamma(b+\zeta\mu+1)$, the only solution of which being $\mu=0$. Hence we find $\mathcal{F}(u)\simeq u^{-1}$, i.e., precisely the form of $\mathcal{F}$ suggested by the experimental data of branched cracks and the numerical data on directed percolation clusters.

We have thus shown how the power law behavior of $N(r)$, $K(r)$, and $\mathcal{F}(u)$ could be simply understood if one assumes that the branched structure is critical in the sense that its average length is just divergent. This scenario occurs in the physically compelling model of two-dimensional directed percolation, but is clearly more general and applies to the cases studied in [26–28]. It would also be very interesting to study other branched cracks to see whether the exponent b is universal or if it can be used to discriminate between different branching modes.

The next step would be, of course, to build a minimal model for the three-dimensional problem. Although we have not found how to generalize Eq. (2) for three-dimensional *branched* cracks, it is interesting to discuss the case of unbranched cracks. The simplest nontrivial equation describing the propagation of a line (the crack front) in a disordered medium is the following:

$$\frac{\partial x}{\partial t}=V+\gamma\frac{\partial^2 x}{\partial y^2}+\frac{\lambda_{xx}}{2}\left(\frac{\partial x}{\partial y}\right)^2+\frac{\lambda_{xz}}{2}\left(\frac{\partial z}{\partial y}\right)^2+\eta_x(y,t)\,, \tag{6a}$$

$$\frac{\partial z}{\partial t}=\gamma\frac{\partial^2 z}{\partial y^2}+\lambda_z\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)+\eta_z(y,t)\,, \tag{6b}$$

where x is the direction of crack propagation, y is along the crack front, and z is the tensile axis. V is the nominal crack velocity, γ is the "line tension," and $\eta_{x,z}$ are noises describing the local disorder in the material. Finally, the nonlinear terms $\{\lambda\}$ describe the fact that the local crack velocity depends on the local direction of the crack, and are the only ones allowed by the symmetries of the problem: $(y\to -y),(z\to -z)$. The trace of the crack front is then the fracture surface, the equation of which is $z(x,y)$ [29]. Interestingly enough, Eq. (6) was recently discussed in the context of driven vortex lines in super-

conductors [30]. One finds that $z(x,y)$ is an *anisotropic* self-affine surface:

$$\langle [z(x,y)-z(0,0)]^2 \rangle = y^{2\zeta} G(x/y^{\phi}) \quad (7)$$

with $G(0)=$const and $G(u) \simeq u^{2\zeta/\phi}$ for large u. Quite surprisingly, the exponents ζ and ϕ are found to depend on the value of the nonlinear couplings $\{\lambda\}$: For Gaussian noises $\eta_{x,z}$, Ertas and Kardar [30] find $\zeta \simeq 0.5$ and $\phi \simeq 1.5$–2.0 in most regions of the parameter space $\{\lambda\}$, but $\zeta \simeq 0.75$, $\phi \simeq 1.5$ when $\lambda_{xz} \ll \lambda_{xx}, \lambda_z$. The latter result is quite close to the experimental values [1,3,5–7] $\zeta \simeq 0.8$. However, no significant anisotropy was detected in the 2D cuts of Ref. [3], suggesting rather $\phi \simeq 1$ (although one could imagine mechanisms by which isotropy is restored on average). Even if this coincidence is intriguing, we feel that more work on the very rich structure of Eq. (6) is needed to argue convincingly that these equations may offer a key to the fractal nature of unbranched cracks. We believe, however, that Eq. (6) is at least of pedagogical value, since it shows how a reasonable equation for crack propagation naturally leads to self-affine fracture surfaces.

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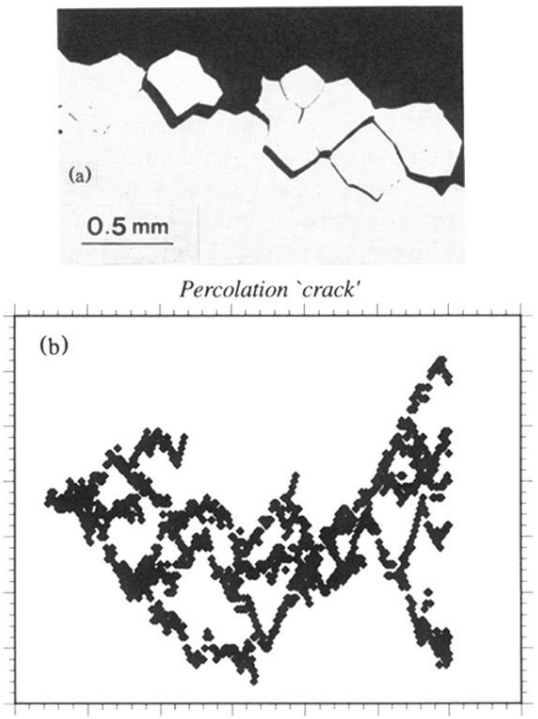

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