Partial Dynamical Symmetry and the Suppression of Chaos

N. Whelan* and Y. Alhassid
Center for Theoretical Physics, Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06511

A. Leviatan
Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
(Received 9 April 1993)

Partial dynamical symmetry is a situation in which the Hamiltonian does not have a certain symmetry yet a subset of its eigenstates does. It is shown that partial dynamical symmetry may cause suppression of chaos even in cases where the fraction of states which has the symmetry vanishes in the classical limit. The average entropy associated with the symmetry is a sensitive quantum measure of the partial symmetry and its effect on the chaotic dynamics.

PACS numbers: 05.45.+b

The dynamics of a generic classical Hamiltonian system is mixed [1]; KAM islands of regular motion and chaotic regions coexist in phase space. In the associated quantum system it is usually difficult to separate the regular states from the irregular ones, although techniques based on torus quantization and discrete symmetries have had some success [1,2]. If no separation between regular and irregular states is done, the statistical properties of the spectrum are usually intermediate between the Poisson and the Gaussian orthogonal ensemble (GOE) statistics. Even when the system seems to be fully chaotic, regular states may exist. This is the case of the hydrogen atom in strong magnetic field near threshold where a series of regular quasi Landau resonances coexist in a region of otherwise chaotic dynamics [3]. These resonances can be explained by WKB quantization [4] and are associated with approximate symmetries [5].

It is well known that symmetry properties may have a large influence on the purity of eigenstates. The symmetry in question is often fulfilled by only a subset of eigenstates. Such partial symmetries occur, for example, in discrete nuclear states embedded in a continuum of decay channels [6]. On the other hand, Hamiltonians with dynamical symmetry are always completely integrable [7,8]. A dynamical symmetry is a situation where the Hamiltonian can be written as a function of the Casimir invariants of a chain of subalgebras:

$$G_0 \supset \cdots \supset G \supset \cdots \supset G' . \tag{1}$$

The Casimir invariants, together with invariants associated with missing labels, form a complete set of constants of the motion in involution. The eigenstates are labeled by the irreducible representations (irreps) of the algebras in the chain and the eigenvalues are known analytically. The classical motion is purely regular.

Recently, the concept of dynamical symmetry has been generalized [9] to include situations in which only a subset of the Hamiltonian eigenstates is solvable, has good symmetry, and can be labeled by irreps of G. The Hamiltonian itself is not invariant under G and various irreps are mixed in all the other eigenstates. Such a situation is referred to as partial dynamical symmetry. The Casimir invariants of G do not commute with H and are no longer constants of the motion. The system is not completely integrable and may exhibit chaotic behavior.

The work of Ref. [9] established a concrete algorithm to construct Hamiltonians with partial symmetries. Unlike the situation in Ref. [5], the symmetry of the solvable states is exact, yet does not arise from invariance properties of the Hamiltonian. This offers an important opportunity to study how the existence of partial (but exact) symmetries affects the dynamics of the system. If the fraction of solvable states remains finite in the classical limit, one might expect that a corresponding fraction of the phase space would consist of KAM tori and exhibit regular motion. The purpose of this paper is to point out that partial dynamical symmetry can have an even greater effect on the dynamics. We present an example in which partial dynamical symmetry is strongly correlated with suppression (i.e., reduction) of chaos even though the fraction of solvable states approaches zero in the classical limit.

A Hamiltonian which possesses a partial SU(3) dynamical symmetry was presented in Ref. [9] in the framework of the interacting boson model of nuclei [10]. The degrees of freedom of the model are one monopole boson ($s^\dagger$) and five quadrupole bosons ($d_\mu^\dagger$). The bilinear combinations $\{s^\dagger s, s^\dagger d_\mu, d_\mu^\dagger s, d_\mu^\dagger d_\nu\}$ form a U(6) algebra. This model has been used in analyzing spectroscopic properties of quadrupole collective states in a wide range of nuclei. Within the model we consider the following family of Hamiltonians [9,11]:

$$H = h_0 P_0^\dagger P_0 + h_2 \sum_\mu P_{2\mu}^\dagger P_{2\mu} , \tag{2}$$

where

$$\begin{aligned} P_0^\dagger &= d^\dagger \cdot d^\dagger - \beta_0^2 (s^\dagger)^2 , \\ P_{2\mu}^\dagger &= \beta_0 s^\dagger d_\mu^\dagger + \sqrt{7/2}(d^\dagger \times d^\dagger)_\mu^{(2)} \end{aligned} \tag{3}$$

are boson pair operators of angular momentum $L=0$ and 2, respectively, and standard notation for a tensor product is used [12]. h_0, h_2, and β_0 are parameters. When $\beta_0=\sqrt{2}$, the Hamiltonian (2) reduces to that of Ref. [9] and has an SU(3) partial dynamical symmetry. If in addition $h_2/h_0=2$, the Hamiltonian exhibits an SU(3) dynamical symmetry. Other dynamical symmetries of the Hamiltonian (2) are obtained for $h_2=0$ [O(5) symmetry]; $h_2=0$ and $\beta_0=1$ [O(6) symmetry]; and $\beta_0=0$ or $\beta_0\to\infty$ [U(5) symmetry].

The solvable states corresponding to the partial SU(3) symmetry at $\beta_0=\sqrt{2}$ are obtained by angular momentum projection from the following states with N bosons:

$$|k\rangle \propto (P^\dagger_{2,2})^k[(s^\dagger+\sqrt{2}d^\dagger_0)/\sqrt{3}]^{N-2k}|0\rangle\,. \tag{4}$$

For a given integer $k\geq 0$ the states in (4) are eigenstates of the Hamiltonian (2) with eigenvalues $E_k=3h_2k(2N+1-2k)$. An eigenstate $|k\rangle$ is a lowest weight state in the SU(3) representation $(2N-4k,2k)$. The states in (4) do not have good angular momentum L but do have well defined angular momentum projection K along the symmetry z axis. They represent intrinsic states of the ground band ($k=0$) and excited γ-vibrational bands ($k>0$) of a prolate nucleus. States of good angular momentum $|(2N-4,2k);K=2k,L,M\rangle$ can be projected from the states in (4). Since the projection operator is constructed from O(3) generators, it commutes with the rotation invariant Hamiltonian in (2) so that the projected states are also eigenstates of H when $\beta_0=\sqrt{2}$. Furthermore, since O(3) is a subgroup of SU(3), these projected eigenstates retain the SU(3) character of the parent intrinsic state (4).

The number of solvable states for a given spin L and boson number N is

$$\begin{aligned} &([L/2]+\eta)(2L+1), \quad 0\leq L\leq N\,,\\ &\{[(2N-L)/2]+\eta\}(2L+1), \quad N\leq L\leq 2N\,, \end{aligned} \tag{5}$$

with $\eta=1$ for L even and $\eta=0$ for L odd. $[x]$ is the integer smaller than or equal to x. There are no states in the model with $L>2N$, and the $2L+1$ factor in (5) arises from the usual M multiplicity for a given L. It is also possible to estimate [13] the total number of states for a given choice of N and L.

To leading order in $1/N$ the fraction f of solvable states as a function of the spin per boson l is

$$f(N,l)=F(l)/N^2\,, \tag{6}$$

where

$$F(l)=\begin{cases}144/(12-18l+7l^2), & 0\leq l\leq 1\,,\\ 144/(2-l)^2, & 1\leq l\leq 2\,.\end{cases} \tag{7}$$

Thus, at a given l, the fraction of solvable states decreases like $1/N^2$ with boson number. However, at a given boson number N, this fraction increases with l. We emphasize that Eqs. (6) and (7) are valid only in the limit of very large N. However, the feature that $F(l)$ increases with l is true even for finite N.

The classical limit of (2) is obtained [8,14,15] through the use of coherent states parametrized by the six complex numbers $\{\alpha_s,\alpha_\mu;\mu=-2,\dots,2\}$ and taking $N\to\infty$. The classical Hamiltonian is then obtained from (2) and (3) by the substitution $s^\dagger,d^\dagger_\mu\to\alpha^*_s,\alpha^*_\mu$ and $s,d_\mu\to\alpha_s,\alpha_\mu$ and rescaling the parameters $h_i\to Nh_i$ $(i=0,2)$. Notice that $1/N$ plays the role of $\hbar$.

To study the effect of the SU(3) partial dynamical symmetry on the dynamics, we fix the ratio h_2/h_0 at a value far from the exact SU(3) symmetry (for which $h_2/h_0=2$) and far from the other symmetry limits of the Hamiltonian. We then change β_0 in the range $1\leq\beta_0\leq 2$. Classically, we determine the fraction σ of chaotic volume and the average largest Lyapunov exponent [14] $\bar{\lambda}$. To analyze the quantum Hamiltonian, we study spectral and transition intensity distributions [16]. The nearest neighbors level spacing distribution is fitted by a Brody distribution $P_\omega(S)=AS^\omega\exp(-\alpha S^{1+\omega})$, where A and α are determined by the conditions that $P_\omega(S)$ is normalized to 1 and $\langle S\rangle=1$. For the Poisson statistics $\omega=0$ and for the GOE $\omega=1$, corresponding to integrable and fully chaotic classical motion [17,18], respectively. The intensity distribution of the SU(3) quadrupole ($E2$) operator,

$$T^{(2)}_\mu \propto (d^\dagger\times s+s^\dagger\times\tilde{d})^{(2)}_\mu-\frac{\sqrt{7}}{2}(d^\dagger\times\tilde{d})^{(2)}_\mu\,, \tag{8}$$

is fitted by a χ^2 distribution in ν degrees of freedom [19],

$$P_\nu(y)=[(\nu/2\langle y\rangle)^{\nu/2}/\Gamma(\nu/2)]y^{\nu/2-1}\exp(-\nu y/2\langle y\rangle)\,.$$

For the GOE, $\nu=1$ and ν decreases as the dynamics becomes more regular [20].

Figure 1 shows the two classical measures $\sigma,\bar{\lambda}$ and the two quantum measures ω,ν for the Hamiltonian (2) as a function of β_0. The parameters of the Hamiltonian are taken to be $h_2/h_0=15$ and the number of bosons is $N=25$. Shown are three classical spins $l=0.08$, 0.4, and 1 which correspond in the quantum case to $L=2$, 10, and 25. All measures show a pronounced minimum which gets deeper and closer to $\beta_0=\sqrt{2}$ [where the partial SU(3) symmetry occurs] as the classical spin increases. This behavior is correlated with the fraction of solvable states (at a constant N) being larger at higher l. The strong dependence on l seen in Fig. 1 is in contrast to the weak dependence of chaos on l (for $l\leq 1$) observed in other regions of the interacting boson model [14,16] parameter space. We remark that the classical measures show a clear enhancement of regular motion near $\beta_0=\sqrt{2}$ even though the fraction of solvable states vanishes as $1/N^2$ in the classical limit $N\to\infty$.

To confirm that the observed suppression of chaos is related to the SU(3) partial dynamical symmetry we employ the concept of an entropy [21,22] associated with a given symmetry. To determine the SU(3) entropy, we expand any eigenstate $|\alpha LM\rangle$ in an SU(3) basis $|(\lambda,\mu)KLM\rangle$,

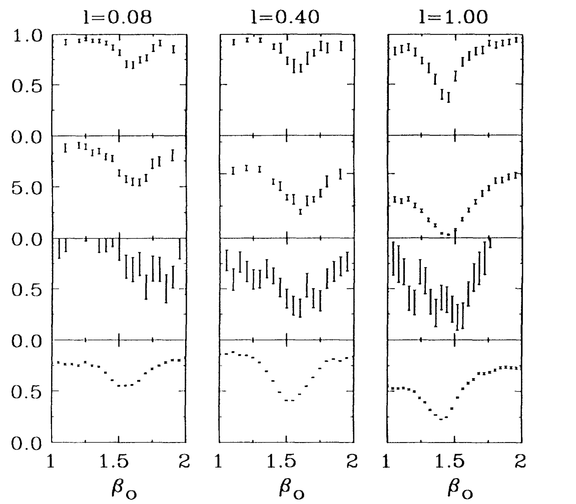

FIG. 1. Classical $(\sigma,\bar{\lambda})$ and quantal (ω,ν) measures of chaos versus β_0 for the Hamiltonian (2) with $h_2/h_0=15$. Shown are three cases with classical spins $l=0.08$, 0.4, and 1. The quantal calculations (ω,ν) are done for $N=25$ bosons and spins $L=2$, 10, and 25, respectively. Notice that with increasing spin the minimum gets deeper and closer to $\beta_0=\sqrt{2}$. The suppression of chaos near $\beta_0=\sqrt{2}$ is seen both for finite N through the measures ω,ν and in the classical limit $N\to\infty$ through the measures $\sigma,\bar{\lambda}$.

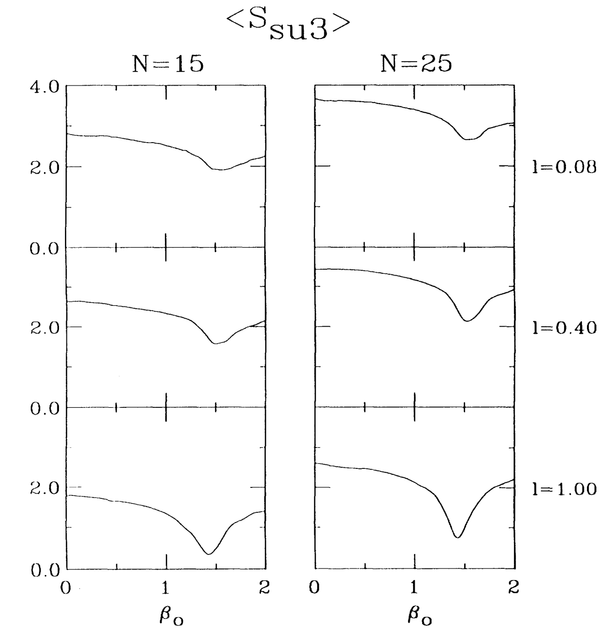

FIG. 2. The average SU(3) entropy of the eigenstates of the Hamiltonian (2) (for $h_2/h_0=15$) versus β_0, for three values of the spin (per boson), $l=0.08$, 0.4, and 1. Left: $N=15$ bosons; right: $N=25$ bosons.

$$|\alpha LM\rangle = \sum_{(\lambda,\mu),K} c^{(\alpha)}_{(\lambda,\mu)K}\,|(\lambda,\mu)KLM\rangle\,. \tag{9}$$

Denoting by $p^{(\alpha)}_{\lambda\mu}$ the probability to be in the SU(3) irrep (λ,μ)

$$p^{(\alpha)}_{\lambda\mu} = \sum_K |c^{(\alpha)}_{(\lambda\mu)K}|^2\,, \tag{10}$$

the SU(3) entropy of the state $|\alpha LM\rangle$ is defined as

$$S^{(\alpha)}_{\mathrm{SU}(3)} = -\sum_{\lambda,\mu} p^{(\alpha)}_{\lambda\mu}\ln p^{(\alpha)}_{\lambda\mu}\,. \tag{11}$$

The entropy vanishes when the state has a good SU(3) symmetry. The averaged entropy $\langle S_{\mathrm{SU}(3)}\rangle$ over all eigenstates is then a measure of the global SU(3) symmetry. This quantity is plotted in Fig. 2 versus β_0 for $N=15$ and 25 and for the same spin values (per boson) l as in Fig. 1. We observe a minimum which is well correlated with the minimum in Fig. 1. The maximum SU(3) entropy is the logarithm of the number of allowed SU(3) irreps for the given N and L. The average SU(3) entropy therefore increases with N. The depth of the minimum increases with N and l although the fraction of solvable states is smaller at $N=25$ than at $N=15$ by a factor of about 3. The existence of an SU(3) partial dynamical symmetry seems to have an effect of increasing the SU(3) symmetry of all the states, not just those with an exact SU(3) symmetry.

To summarize, we have considered the effect of partial symmetry of a Hamiltonian on the nature of the underlying dynamics. The symmetry under consideration is exact but is shared by only some of the quantum eigenstates. We have demonstrated that partial dynamical symmetry can suppress chaos even when the fraction of exactly solvable states vanishes in the classical limit. In general, most systems exhibit coexistence of regular and irregular motion. Hamiltonians with partial symmetries can play an important role in understanding this coexistence as well as shed light on the important problem of the influence of a symmetry on the interplay between order and chaos in dynamical systems.

This work was supported in part by the U.S. Department of Energy Grant No. DE-FG-0291-ER-40608 and by the Israeli Science Ministry. A.L. thanks the Center for Theoretical Physics at Yale University for their hospitality.

*Present address: The Niels Bohr Institute, DK-2100 Copenhagen, Denmark.

[1] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rev. Lett. **64**, 1479 (1990); Phys. Rep. **223**, 43 (1993).
[2] I. C. Percival, J. Phys. B **6**, L229 (1973).
[3] H. Friedrich and D. Wintgen, Phys. Rep. **183**, 37 (1989).
[4] J. A. C. Gallas and R. F. O'Connell, J. Phys. B **15**, L75 (1982).
[5] D. Delande and J. C. Gay, Phys. Rev. Lett. **59**, 1809 (1987).
[6] I. Rotter, Rep. Prog. Phys. **54**, 635 (1991).
[7] W. Zhang, C. C. Martens, D. H. Feng, and J. Yuan,

Phys. Rev. Lett. **61**, 2167 (1988).
[8] Y. Alhassid, A. Novoselsky, and N. Whelan, Phys. Rev. Lett. **65**, 2971 (1990).
[9] Y. Alhassid and A. Leviatan, J. Phys. A **25**, L1265 (1992).
[10] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge Univ. Press, Cambridge, 1987).
[11] A. Leviatan, Ann. Phys. (N.Y.) **179**, 201 (1987).
[12] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, NJ, 1957).
[13] N. Whelan, Ph.D. thesis, Yale University, 1993.
[14] Y. Alhassid and N. Whelan, Phys. Rev. C **43**, 2637 (1991).
[15] R. L. Hatch and S. Levit, Phys. Rev. C **25**, 614 (1982).
[16] Y. Alhassid and A. Novoselsky, Phys. Rev. C **45**, 1677 (1992); N. Whelan and Y. Alhassid, Nucl. Phys. **A556**, 42 (1993).
[17] M. V. Berry and M. Tabor, Proc. R. Soc. London A **356**, 375 (1977).
[18] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[19] Y. Alhassid and R. D. Levine, Phys. Rev. Lett. **57**, 2879 (1986).
[20] Y. Alhassid and M. Feingold, Phys. Rev. A **39**, 374 (1989).
[21] F. Iachello and R. D. Levine, Europhys. Lett. **4**, 389 (1987).
[22] F. M. Izrailev, J. Phys. A **22**, 865 (1989).