

Observing the Inflaton Potential

Edmund J. Copeland,¹ Edward W. Kolb,^{2,3} Andrew R. Liddle,⁴ and James E. Lidsey²

¹*School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, United Kingdom*

²*NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois 60510*

³*Department of Astronomy and Astrophysics, The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637*

⁴*Astronomy Centre, School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, United Kingdom*
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We show how observations of the density perturbation (scalar) spectrum and the gravitational wave (tensor) spectrum allow a reconstruction of the potential responsible for cosmological inflation. A complete functional reconstruction or a perturbative approximation about a single scale are possible; the suitability of each approach depends on the data available. Consistency equations between the scalar and tensor spectra are derived, which provide a powerful signal of inflation.

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One of the most exciting aspects of the recent detection of large angle microwave background anisotropies by the Cosmic Background Explorer satellite (COBE) [1] is the possibility that part of the anisotropy observed is due to long wavelength gravitational wave (tensor) modes instead of (scalar) density perturbations. In general the influence of scalar and tensor modes on microwave background anisotropies differs as a function of angular scale, and the use of measurements on different scales may allow one to separate the anisotropies into their scalar and tensor components. This has recently been considered by Crittenden *et al.* [2].

This prospect is especially exciting for models of cosmological inflation, proposed over a decade ago [3] as a possible resolution of a number of otherwise puzzling aspects of the standard hot big bang cosmology [4]. Inflation has long been known to predict that both scalar modes [5] and tensor modes [6–8] should exist on all astrophysically relevant scales. Although the generic prediction from inflation has in the past advertised as a flat (Harrison-Zel'dovich) scalar spectrum and a tensor spectrum of negligible amplitude, the rapid improvement of observational data has led many researchers [9] to emphasize recently the importance of taking the detailed inflationary predictions seriously. Typically the predictions from inflation are that the scalar spectrum possesses a scale *dependence*, which is weak in many models but can be rather marked in others. And though the amplitude of tensors may typically be less than that of the scalars, this does not necessarily imply that it is negligible.

This is problematic from a large-scale-structure viewpoint, since different inflationary models offer a range of predictions and there is currently no clear guidance from particle physics as to which inflationary models may be suitable. The correct input one should make into a large scale structure model is therefore unknown. However, from an inflationary viewpoint this is a promising feature, as it raises the possibility that improved observations may allow one to distinguish between inflationary models. The aim of this Letter is to investigate the use of observations precisely to this end, by deriving equations which al-

low one to proceed from a knowledge of the scalar and/or tensor spectrum to a determination of the inflaton potential. As a very useful by-product, we derive a consistency relation between the allowed scale dependences of the scalar and tensor modes. This is a powerful discriminant for inflationary models in general, as it does not depend on a specific choice of inflationary model.

Reconstruction of the inflaton potential in this manner was first considered by Hodges and Blumenthal [10] (hereafter HB). We improve upon their results in two important ways. First, we consider both scalar and tensor modes, whereas they restricted their study to the scalars alone. This is a vital improvement, because, as HB acknowledged and we rederive, the scalars alone are insufficient to uniquely determine the inflaton potential—such a reconstruction is possible only up to an undetermined constant, and as the reconstruction equations are nonlinear this leads to functionally different potentials giving rise to the same spectrum. The tensors (even just the tensor amplitude at a single scale) provide just the extra information needed to lift this degeneracy. Second, their analysis made explicit use of the so-called *slow-roll* approximation. It is well known that this approximation breaks down unless both the scalar spectrum is nearly flat and the tensor amplitude is negligible. We consider the inflation dynamics in full generality. However, general expressions for the perturbation spectra are not known, and one must use the slow-roll approximation there. It is shown in another paper [11] that this hybrid approach offers substantial improvements over pure slow-roll results.

The equations of motion are most conveniently written in the $H(\phi)$ formalism [12]. An isotropic scalar field ϕ in a spatially flat universe satisfies

$$(H')^2 - \frac{3}{2} \kappa^2 H^2 = -\frac{1}{2} \kappa^4 V(\phi), \quad (1)$$

$$\kappa^2 \ddot{\phi} = -2H', \quad (2)$$

provided that $\dot{\phi}$ does not pass through zero, where over-dots are time derivatives, primes are derivatives with respect to ϕ , and $\kappa^2 = 8\pi G = 8\pi/m_{\text{Pl}}^2$. The usual slow-roll

approximation amounts to neglecting the first term in Eq. (1) and its ϕ derivative.

The amplitudes of the scalar and tensor modes may be written using the standard expressions as

$$A_S(\phi) = \frac{\sqrt{2}\kappa^2}{8\pi^{3/2}} \frac{H^2(\phi)}{|H'(\phi)|}, \quad (3)$$

$$A_G(\phi) = \frac{\kappa}{4\pi^{3/2}} H(\phi), \quad (4)$$

respectively. A_S is equivalent to $P^{1/2}(k)/3\sqrt{2}\pi$ in HB, to δ_H of Ref. [13], and for a flat spectrum equal to $4\pi\epsilon_H$ of Ref. [7]. A_G^2 is equivalent to $\mathcal{P}_g/32\pi$ of Ref. [13]. One immediately notes that

$$\frac{A_G}{A_S} = \frac{\sqrt{2}}{\kappa} \frac{|H'|}{H} = \frac{\sqrt{2}}{\kappa} \left| \frac{d \ln A_G}{d\phi} \right|, \quad (5)$$

so the inflationary condition $\ddot{a} > 0$ implies $A_G < A_S$. However, the relative contribution of tensors to scalars for large angle microwave background anisotropies is given roughly (for sufficiently flat spectra) by the ratio $25A_G^2/2A_S^2$ [11], so it is possible for the tensor contribution to dominate the anisotropy.

The spectra are quoted above as functions of ϕ —that is, we are given the amplitude when the scalar field takes a particular value. To compare with observations we must relate ϕ to a given cosmological scale λ . This is achieved by utilizing the formula

$$N(\phi) \equiv \int_{t_e}^t H(t') dt' = -\frac{\kappa^2}{2} \int_{\phi}^{\phi_e} \frac{H(\phi')}{H'(\phi')} d\phi', \quad (6)$$

which gives the number of e -foldings between a scalar field value ϕ and the end of inflation at $\phi = \phi_e$. Each length scale λ is associated with a unique value of ϕ when that scale crossed the Hubble radius during inflation, indicated by writing $\lambda(\phi)$. That value of ϕ is also associated with a value $a(\phi)$ of the scale factor. We can make use of Eq. (6) to relate $a(\phi)$ to the value of the scale factor at the end of inflation, a_e : $a(\phi) = a_e \exp[-N(\phi)]$, which allows us to express $\lambda(\phi)$ as

$$\lambda(\phi) = \frac{\exp[N(\phi)]}{H(\phi)} \frac{a_0}{a_e}. \quad (7)$$

Differentiating Eq. (7) with respect to ϕ yields

$$\frac{d\lambda(\phi)}{d\phi} = \pm \frac{\kappa}{\sqrt{2}} \left[\frac{A_S}{A_G} - \frac{A_G}{A_S} \right] \lambda. \quad (8)$$

Note that the reconstruction equation derived by HB [their Eq. (2.10)] has only the first term on the right-hand side of Eq. (8), indicating their assumption of slow-roll behavior (which here amounts to neglecting terms of order A_G^2/A_S^2).

Substituting Eq. (8) into Eq. (5) gives

$$\frac{\lambda}{A_G(\lambda)} \frac{dA_G(\lambda)}{d\lambda} = \frac{A_G^2(\lambda)}{A_S^2(\lambda) - A_G^2(\lambda)}. \quad (9)$$

This is a very important equation, because it is valid for any inflation potential and indicates a strong connection between the forms of the scalar and tensor spectra produced by inflation. The left-hand side is essentially just half of the (scale-dependent) spectral index of the tensor spectrum. Potentially, this provides a powerful discriminator as to the correctness of inflation. We shall refer to it as the *consistency equation*.

It highlights the asymmetry in the correspondence between the scalar and tensor spectra. If one were given the tensor spectrum, then a simple differentiation supplies the unique scalar spectrum. However, if a scalar spectrum is supplied, then this first-order differential equation must be solved to find the form of $A_G(\lambda)$. This leaves an undetermined constant in the tensor spectrum and, as the consistency equation is nonlinear, this implies that the scalar spectrum alone does not uniquely specify the functional form of the tensors. However, knowledge of the amplitude of the tensor spectrum at one scale is sufficient to determine this constant and lift the degeneracy.

It is the tensor spectrum one requires to proceed with reconstruction. Once the form of the tensor spectrum has been obtained, either directly from observation or by integrating Eq. (9), the potential, as parametrized by λ , may be derived by substituting Eqs. (3) and (4) into Eq. (1). This gives

$$V[\phi(\lambda)] = \frac{16\pi^3 A_G^2(\lambda)}{\kappa^4} \left[3 - \frac{A_G^2(\lambda)}{A_S^2(\lambda)} \right], \quad (10)$$

where the final term in the square brackets again improves on HB. Finally, integration of Eq. (8) yields the function $\phi = \phi(\lambda)$ as

$$\begin{aligned} \phi(\lambda) &= \pm \frac{\sqrt{2}}{\kappa} \int^{\lambda} \frac{d\lambda'}{\lambda'} \frac{A_S(\lambda') A_G(\lambda')}{A_S^2(\lambda') - A_G^2(\lambda')} \\ &= \pm \frac{\sqrt{2}}{\kappa} \int^{A_G} dA_G' \frac{A_S[A_G']}{A_G'^2}, \end{aligned} \quad (11)$$

where we have absorbed the integration constant by taking advantage of the freedom to shift ϕ by a constant. The second integral follows after substitution of the consistency equation and is appropriate if the functional form of A_S as a function of A_G is known. The functional form of $V(\phi)$ follows by inverting Eq. (11) and substituting the result into Eq. (10).

The reconstruction equations are Eqs. (9), (10), and (11). We emphasize again that even an arbitrarily accurate determination of the scalar spectrum will not allow one to determine the inflaton potential—at least a minimal knowledge of the tensors is required. Ultimately, though, one might hope to overdetermine the problem by having observational knowledge of both spectra over a range of scales. The consistency equation (9) must then be satisfied, or the inflationary hypothesis has been disproved (up to the accuracy of the slow-roll approximation for the perturbation spectra).

The reconstruction equations allow a functional reconstruction of the inflaton potential. For suitably simple spectra, this can be done analytically, and in another paper [11] we illustrate this for the well-known cases of scalar spectra which are exactly scale invariant, logarithmically corrected from scale invariance and exact power laws. The earliest observations with an accuracy useful for our purposes are likely to only provide such simple functional fits. For advanced observations, however, one might expect that the reconstruction equations would have to be solved numerically. There are additional issues related to observational errors which we do not investigate here (but see Ref. [11]).

An alternative approach, useful for obtaining mass scales, is to concentrate on data around a given length scale λ_0 , and perturbatively derive the potential around its corresponding scalar field value $\phi_0 \equiv \phi(\lambda_0)$. If we know $A_G(\lambda_0)$ and $A_S(\lambda_0)$ separately, then $V(\phi_0)$ follows immediately from Eq. (10). In order to make further progress, one also needs information regarding the derivatives of the spectra. Of course, the measurement of these derivatives requires knowledge of the spectra over at least a limited range of scales, so this process is equivalent to a Taylor expansion of the functional reconstruction [14].

To obtain $V'(\phi)$, one needs only the derivative of the scalar spectrum, or equivalently its spectral index. This is fortunate, as its tensor equivalent would be much harder to observe. With the scalar spectral index n (in general a function of scale) defined as usual by

$$1 - n = \frac{d \ln A_S^2(\lambda)}{d \ln \lambda}, \quad (12)$$

one can show that

$$V'(\phi_0) \equiv \left. \frac{dV(\phi)}{d\phi} \right|_{\phi=\phi_0} = \pm \frac{16\pi^3}{\sqrt{2}\kappa^3} \frac{A_G^3(\lambda_0)}{A_S(\lambda_0)} \times \left[7 - n_0 - (5 - n_0) \frac{A_G^2(\lambda_0)}{A_S^2(\lambda_0)} \right], \quad (13)$$

where $n(\lambda_0) \equiv n_0$. If one wishes, this can be simplified into the slow-roll approximation (in which $n_0 \approx 1$) by ignoring the final term in the square brackets.

One can continue this process. At no stage is knowledge of the tensor spectrum derivative required, because the consistency equation can always be used to remove it. Given the second derivative of the scalars (equivalently the first derivative of the scalar spectral index), one can derive an expression for $V''(\phi_0)$, quoted in Ref. [11], but it is too cumbersome to reproduce here. Its slow-roll limit does not require n'_0 , and is

$$V''_{sr}(\phi_0) = \frac{4\pi^3}{\kappa^2} \frac{A_G^2(\lambda_0)}{A_S^2(\lambda_0)} [4(n_0 - 4)^2 A_G^2(\lambda_0) - (1 - n_0)(7 - n_0) A_S^2(\lambda_0)]. \quad (14)$$

This offers the prospect of determining whether the

inflaton potential is concave or convex when the presently observable universe crossed outside the Hubble radius during inflation. We note immediately that V'' is positive if $1 < n_0 < 7$. It is the amplitude of the tensor perturbations at a particular scale which yields information regarding the mass scale at which these processes are occurring during inflation. The steepness of the potential, measured by the dimensional parameter $V(\phi_0)/|V'(\phi_0)|$, is determined by the ratio $A_S(\lambda_0)/A_G(\lambda_0)$.

Let us illustrate by example. Within a few years a combination of microwave background anisotropy measurements should give us some information about the scalar and tensor amplitudes at a particular length scale λ_0 (corresponding to an angular scale θ_0) [2]. A hypothetical, but plausible, data set that this might provide would be $A_S(\lambda_0) = 1 \times 10^{-5}$; $A_G(\lambda_0) = 2 \times 10^{-6}$; $n_0 = 0.9$. This would lead to

$$\begin{aligned} V(\phi_0) &= (2 \times 10^{16} \text{ GeV})^4, \\ \pm V'(\phi_0) &= (3 \times 10^{15} \text{ GeV})^3, \\ V''_{sr}(\phi_0) &= (5 \times 10^{13} \text{ GeV})^2. \end{aligned} \quad (15)$$

In this way cosmology might be first to get a "piece of the action" of physics at the grand unification scale.

In this Letter we have discussed the promising possibility of large scale structure observations, particularly of tensor modes, providing rather specific information as regards the physics of the grand unified era. We have derived equations which allow a knowledge of either the scalar spectrum, the tensor spectrum, or preferably both, to be used to reconstruct the potential of the inflaton field. We have also noted a consistency equation, by which the scale dependences of the spectra *must* be related if their origin lies in an inflationary era. This potentially provides a powerful test of inflation; the minimum knowledge required to implement it would be knowledge of the scalar spectrum across a range of scales plus the amplitude of the tensor spectrum at *two* of the wavelengths. [Technically the minimum is the tensor spectrum plus the scalar amplitude at a single scale, but observationally that would be considerably more demanding.]

While the technical aspects of reconstruction are interesting, the practicality of such a program is the main concern. A variety of observations are available to allow this to proceed. The only means likely to yield information on the tensors is large and intermediate angle microwave anisotropy measurements [2]. However, we have emphasized that one only needs a measurement of the tensors at a single scale to reconstruct. For the scalar modes we have the entire slew of large scale structure observations to consider, including the distribution of galaxies, peculiar velocity flows, and the formation times of high redshift objects. There is not enough space to embark on a detailed discussion here, but all these issues, covering both the present and prospective observational

situation, are discussed in depth in another paper [11]. In that paper we also investigate other aspects of this paper in more detail, including examples of functional reconstruction. While the ambitious aim of full reconstruction appears to lie some way into the future, we are optimistic as to the short-term possibilities, as tantalizingly indicated in [2], of obtaining at least a perturbative reconstruction of the inflaton potential and a window on grand unification scale physics.

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