Microwave Background Anisotropy in a Toroidal Universe

Daniel Stevens, Douglas Scott, and Joseph Silk
Departments of Astronomy and Physics, and Center for Particle Astrophysics, University of California, Berkeley, California 94720
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Large-scale cosmic microwave background temperature fluctuations are calculated for a universe with the topology of a 3-torus. In such a universe only perturbations which are harmonics of the fundamental mode are permitted. By comparison with data from the Cosmic Background Explorer satellite, we find that the minimum (comoving) scale of a cubic toroidal universe is $2400h^{-1}$ Mpc for an $n=1$ inflationary model. This is approximately an order of magnitude greater than previous limits and 80% of the horizon scale, implying that a topologically "small" universe is no longer an interesting cosmological model.

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Astrophysical observations demonstrate that to a high degree of accuracy our Universe is homogeneous and isotropic, locally coinciding with the Friedmann model. However, the three-dimensional space of this model may, in general, have a complicated topological structure. Many such multiply connected or "spliced" universes are possible, the simplest being the three-dimensional torus T^3, found by making the following identifications on three-dimensional Euclidean space [1]: $x \equiv x+L_x$, $y \equiv y+L_y$, $z \equiv z+L_z$. When the periods of the multiply connected universe (L_x, L_y, L_z for T^3) are shorter than the horizon scale, the model is referred to as "small." This case of a small universe, which one naively might assume to be the only observationally detectable multiply connected universe, has been investigated by numerous authors [2].

Previous constraints on the minimal scale of a multiply connected universe have generally relied on one of two methods: either one considers the largest known structure and assumes the scale must be larger than this, or one attempts to identify repeated images of a single object. Typically the constraint is given in terms of the parameter R, an average length scale for the small universe such that the volume $V(t)$ of the basic cell scales as $R^3(t)$. For example, the torus above has $R(t) = (L_x L_y L_z)^{1/3}$. Fairall [3] argues that because we can already distinguish structures up to a scale of ~ 500 Mpc, then R must be greater than this. From the failure to identify quasar opposite pairs, Fang and Liu [4] argue that R should be larger than $200h^{-1}$ Mpc (where h is the Hubble's constant H_0 in units of 100 km $s^{-1}Mpc^{-1}$), while Fang and Sato [5] suggest $R \sim 600$ Mpc based on their interpretation of the *alleged* (see [6]) periodicity in quasar redshifts.

However, as with constraints on large-scale density fluctuations, the cosmic microwave background (CMB) proves to set by far the most challenging limits on topology, and, in particular, on the size of a small universe. Early discussions by Zel'dovich [7] note that density perturbations with an arbitrary wave vector, which are admissible for a simply connected universe, cannot arise in a multiply connected universe; that is, the spectrum of perturbations is discrete rather than continuous. However, neither Zel'dovich nor Sokolov and Starobinskii [8], who reiterate this fact, make explicit calculations of the implications for CMB anisotropy. Fang and Houjun [9] do compute the temperature fluctuations in a multiply connected universe, although they consider only a T^1 small universe, i.e., a circle whose circumference is less than the horizon scale. In this Letter we perform these calculations for the more physical case of a general T^3 universe that is not necessarily small. For simplicity, we limit our discussion to the case of a cubic (rather than cuboidal) universe, although the results could be readily generalized. We constrain the possible scale of a toroidal universe by comparison with data from the differential microwave radiometer (DMR) experiment on the Cosmic Background Explorer (COBE) satellite [10]. This limit is only strictly applicable for the standard scale-invariant inflationary power spectrum of fluctuations. But the fact that COBE detects *any* fluctuations on the largest scales provides a general constraint on "small" universes.

If the primeval density fluctuations are adiabatic, the large scale (Sachs-Wolfe) fluctuations in the temperature T of the background radiation in an Einstein–de Sitter universe can be written as [11]

$$\frac{\delta T}{T}(\mathbf{n}) = -\frac{1}{2}\frac{H_0^2}{c^2}\sum_{\mathbf{k}}\frac{\delta_{\mathbf{k}}\exp(iy\mathbf{k}\cdot\mathbf{n})}{k^2}, \qquad (1)$$

where $\mathbf{n}$ denotes a direction on the sky, y is the radius of the decoupling sphere (i.e., $y = 2c/H_0$), $\delta_{\mathbf{k}}$ is the density fluctuation in Fourier space, and the sum is over wave numbers $\mathbf{k}$. We can describe this temperature fluctuation as an expansion in spherical harmonics, i.e., a multipole expansion, by writing $\delta T(\mathbf{n})/T = \sum_{l,m} a_l^m Y_l^m(\mathbf{n})$ with

$$a_l^m = -2\pi i^l \frac{H_0^2}{c^2}\sum_{\mathbf{k}}\frac{\delta_{\mathbf{k}}}{k^2} j_l(ky) Y_l^m(\mathbf{n}), \qquad (2)$$

where j_l is the lth-order spherical Bessel function. A rotationally invariant coefficient is defined by

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$$a_l^2=\sum_m |a_{lm}|^2/(2l+1)\,,$$

such that the average lth multipole of the temperature fluctuation is given by

$$\langle a_l^2\rangle = 16\pi\sum_{\mathbf{k}} \frac{|\delta_{\mathbf{k}}|^2 j_l^2(ky)}{(ky)^4}\,. \quad (3)$$

In a simply connected universe we would change the sum into an integral and find [12]

$$\langle a_l^2\rangle = \langle a_2^2\rangle \frac{\Gamma(l+(n-1)/2)}{\Gamma(l+(5-n)/2)} \frac{\Gamma((9-n)/2)}{\Gamma((3+n)2)}\,, \quad (4)$$

where we have assumed a power law spectrum of perturbations $|\delta_{\mathbf{k}}|^2 \propto k^n$.

However, in the case of a multiply connected universe, not all wave vectors are allowed. In particular, for a given axis of a toroidal universe, only integral multiples of some cutoff wave vector are permitted. For the torus T^3, we allow only wave vectors of the form $\mathbf{k}=(k_x, k_y, k_z)=(p_x k_x^{\rm cut}, p_y k_y^{\rm cut}, p_z k_z^{\rm cut})$, where p_x, p_y, p_z are whole numbers and $k_i^{\rm cut}=2\pi/L_i$. The $p_x=p_y=p_z=0$ mode is not considered as it represents the "dc" component uniform background. To simplify the calculations we consider a torus with a cubic basic cell, i.e., $L_x=L_y=L_z\equiv L$, and we assume a Harrison-Zel'dovich power spectrum, $|\delta_{\mathbf{k}}|^2\propto k$ (i.e., $n=1$). Equation (3) now reads

$$\langle a_l^2\rangle \propto \sum_{p_x}\sum_{p_y}\sum_{p_z} \left(\frac{L}{2\pi yp}\right)^3 j_l^2\left(\frac{2\pi yp}{L}\right)\,, \quad (5)$$

where $p=(p_x^2+p_y^2+p_z^2)^{1/2}$. The lth multipole of the CMB temperature fluctuations is thus, up to an unknown constant, an explicit function of L/y, the ratio of the torus scale to the horizon scale. We restrict ourselves to the $n=1$ power spectrum because it is the natural (although not unique) result of inflation, which has become part of the standard cosmological model. A small universe model could by itself solve the horizon problem [2], thereby removing some of the motivation for inflation, but not if the scale of the universe is greater than the horizon size at recombination, i.e., $L \gtrsim 200h^{-1}$ Mpc (comoving). Note that there is no reason why inflation should not occur in a small universe model [13]. Furthermore, the standard inflationary argument for producing the $n=1$ spectrum is unaffected by the topology [14]; it relies on the weak coupling of the inflaton field and on the "slow rollover" approximation [15], and does not depend on whether the $\mathbf{k}$ modes are continuous or discrete. It is possible, however, that departures from $n=1$ may arise during the inflationary epoch because of boundary conditions on the inflaton field, but this will not affect our main conclusions.

We explicitly calculate this sum for various torus scales. Because the perturbations depend on an unknown constant (the normalization of the power spectrum $|\delta_{\mathbf{k}}|^2$), we can only find how the multipoles scale for a given topology of the universe. The dipole ($l=1$) is expected to be primarily a result of the Earth's motion with respect to the CMB, and therefore it may be ignored. Zel'dovich and Starobinskii [16] assert that in a cubically symmetric toroidal universe, the quadrupole ($l=2$) moment of temperature fluctuations would vanish. Although this is clearly not true for very large tori (i.e., $L\gg y$), we verify this statement for small toroidal universes. Moreover, all low-order multipole moments are suppressed.

The DMR experiment on COBE [10] finds that the observed CMB anisotropies correspond well to a power spectrum of density perturbations with an index $n=1.1 \pm 0.5$ in a simply connected universe. Using Eq. (4), we calculate the allowed range in multipoles that this implies. We match our calculations for the multipoles in a toroidal universe to the $l=20$ COBE multipole, expecting that (i) at this scale COBE is most correctly measuring the true fluctuations of the temperature of the CMB (i.e., cosmic variance is small); (ii) because the higher multipoles are primarily produced from smaller scale density perturbations, they would be least affected in a multiply connected universe and consequently our figures will clearly illustrate any suppression of the low-order multipoles; and (iii) the resultant constraint on torus size is a conservative limit. The results are plotted in Figs. 1–4. Note that previous constraints on the minimum scale of a small universe, of order $500h^{-1}$ Mpc, corresponds to a torus scale of about 15% of the horizon size, i.e, $L/y=0.15$, as shown in Fig. 1. We find that a T^3 universe of this scale does not agree with the COBE observations; rather, by comparing our $l\geq 2$ multipole re-

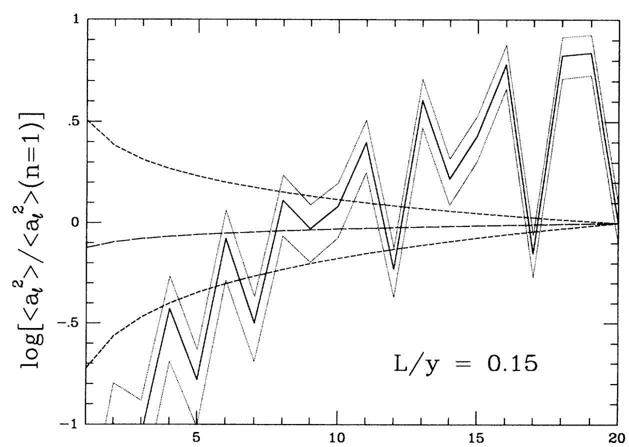

FIG. 1. The scaling of the multipole moments is shown (solid line) for a toroidal universe with torus scale 0.15 times the horizon scale ($L/y=0.15$). One sigma cosmic variance error (dotted lines) is added assuming that the lth multipole has a χ^2_{2l+1} distribution. Dashed lines show the spectrum detected by the COBE DMR experiment, i.e., $n=1.1\pm 0.5$, matched at $l=20$. All values are normalized to a Harrison-Zel'dovich scale-invariant power spectrum ($n=1$). Note that this small toroidal universe, corresponding to the previous minimum scale of a toroidal universe, is incompatible with COBE.

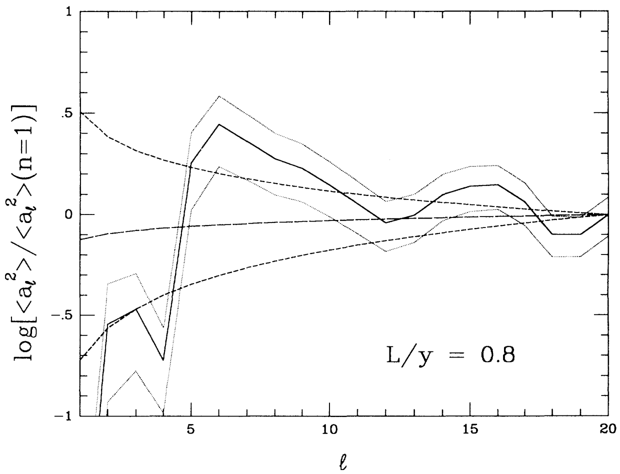

FIG. 2. As in Fig. 1, with torus scale 0.8 times the horizon scale. This is our new minimum scale for a toroidal universe.

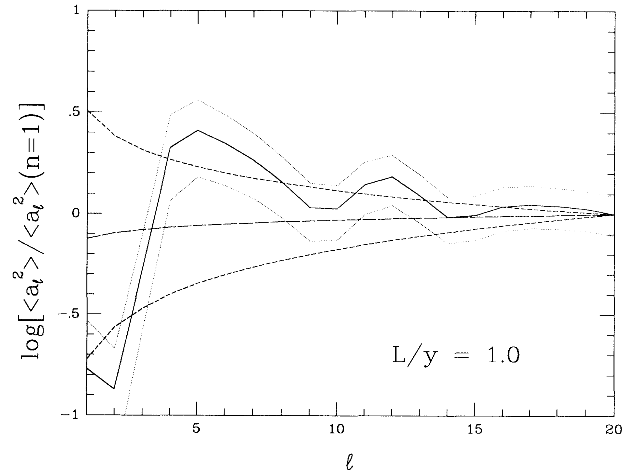

FIG. 3. As in Fig. 1, with torus scale 1.0 times the horizon scale. This is the largest possible "small" universe with T^3 topology. Note the suppression of the quadrupole ($l=2$) moment.

sults for a T^3 universe to COBE observations and performing a Monte Carlo test, we set a new minimal scale of $2400h^{-1}$ Mpc at the 95% confidence level (Fig. 2). For comparison, the horizon size is $3000h^{-1}$ Mpc. This rather conservative limit is expected to become stricter as the COBE measurements improve and are released. We also note that for a limiting small universe model, i.e., $L/y=1.0$ (Fig. 3), there is still a strong suppression of the quadrupole moment. Therefore, an observation of the octopole moment of the CMB temperature fluctuations would produce a strong constraint on the small universe model. For completeness, we show that as $L/y \to \infty$, the torus is indistinguishable from a simply connected universe (Fig. 4).

Although we have restricted ourselves to the case where the three orthogonal scales are the same, there are still a number of ways of identifying the faces of a cube. In fact, there are only four distinct orientable ways to do this [17]. The other three ways lead to somewhat different sums in Eq. (5), but otherwise the calculations can be carried out as before. Specifically, a 180° rotation in the identification of two opposite sides of the cube would require odd numbers of half-integer wavelengths in that direction. For the yz faces rotated, the sum remains as above with the new expression $p^2=[(2p_x-1)/2]^2+p_y^2+p_z^2$. With all three pairs rotated by 180°, we require

$$p^2=\left(\frac{2p_x-1}{2}\right)^2+\left(\frac{2p_y-1}{2}\right)^2+\left(\frac{2p_z-1}{2}\right)^2 .$$

For a 90° rotation (of the xy faces) we identify the k_x and k_y components, so that $p^2=2p_x^2+p_z^2$. For completeness we mention that there are only two further orientable multiply connected topologies for a flat universe [18], consisting of a right regular hexagonal prism rotated by 60° and 120° as the basic cell. Calculations have been made for the above cubic-celled topologies and similarly constrained by the COBE DMR measurements at the 95% confidence level. For the first two topologies we expect a slightly weaker constraint; we find that for the single 180° rotation the limit is $L > 1600h^{-1}$ Mpc, while the triple 180° rotation requires $L > 2900h^{-1}$ Mpc. For the 90° rotation we find that no scale of cubic cell is admissible. Small torus scales, $L \lesssim y$, are ruled out due to the suppression of low order multipoles, while for large torus scales, the multipole spectrum approaches that of an $n=0$ simply connected universe. This asymptotic limit results from the identification of perturbations in two spatial dimensions effectively lowering the degree of the torus by 1. Therefore, as $L/y \to \infty$, the scaling of multipoles approaches that of an $n=1$ simply connected *two-dimensional* universe, which scales identically as the $n=0$ simply connected three-dimensional universe [19].

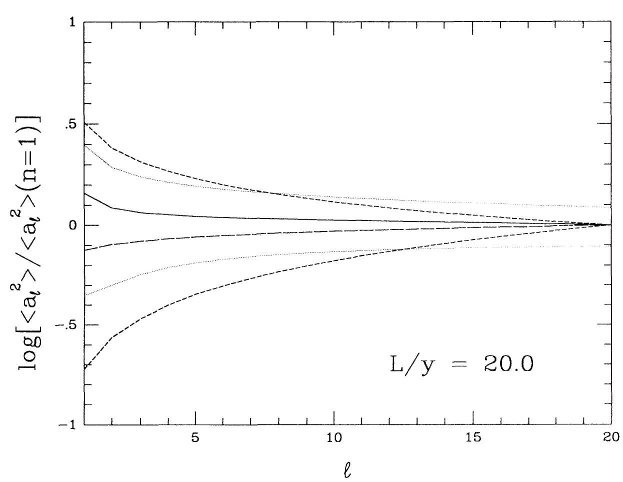

FIG. 4. As in Fig. 1, with torus scale 20 times the horizon scale. Note that for large L/y, the toroidal universe multipole spectrum approaches that of a simply connected flat universe.

Calculations for the hexagonal prism cell would be unnecessarily complicated, but we expect results similar to the third topology above, since we have to mix components from orthogonal directions.

In summary, we have shown that a multiply connected T^3 universe with cell size less than 80% of the present horizon scale is incompatible with the COBE measurements of approximately scale-invariant fluctuations in the cosmic microwave background. Furthermore, reduction of the cell size in a multiply connected universe below approximately the horizon scale results in a quenching of the quadrupole and low multipole anisotropies. Our quantitative conclusion is only strictly valid for the $n=1$ power spectrum generically produced by inflation. However, this model has become the standard cosmological paradigm, and similar conclusions could be obtained for other power laws. Furthermore, the fact that COBE detects *any* fluctuations at the largest scales provides generic constraints on "small" universes. It should be noted that there will always be a perverse density perturbation spectrum in a small universe which could mimic an $n=1$ simply connected CMB fluctuation spectrum, but we do not consider this to be a natural solution. Finally, whereas the limit we have set is very conservative and likely to improve as CMB observations become more exact and are released, we believe that the "small" universe (i.e., cell size less than present horizon scale) has now been ruled out as an interesting cosmological model [20].

[1] Note that the torus T^3 has the same periodic boundary conditions as the cuboid familiar to N-body cosmologists.

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