Edge Voltages and Distributed Currents in the Quantum Hall Effect

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It is shown from rather general assumptions that the current density at low temperatures in the interior of a quantum Hall device follows the electric field distribution, smeared out over an appropriate localization length. This is not in conflict with Laughlin's result that the total Hall current depends only on the potential difference between the edges, but must be used to complement it. It is illustrated by a discussion of edge and bulk currents for a Hall bar under the conditions of the integer quantum Hall effect. It is argued that these results can be extended to the fractional quantum Hall effect.

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Two different approaches have been adopted to explain the quantum Hall effect. In one approach linear response theory is used to show that the Hall current is proportional to the potential gradient, and the constant of proportionality is an integer (or fractional) multiple of e^2/h [1–4]. In the other approach it is shown that the effect of an imbalance in the electrochemical potential between the two edges produces an edge current proportional to the potential difference, again with the same constant of proportionality [5–7]. Both these arguments are correct, and one or both of them can be relevant in particular situations. In this paper I show, on the basis of rather general assumptions, that in the interior of a quantum Hall system there is a current density proportional to the electrostatic field, smeared over a distance equal to the localization distance at the Fermi energy. This result is illustrated by a study of the current distribution in an idealized Hall bar which carries imposed edge currents. These extra electrons that carry the edge currents produce an electrostatic field in the interior of the bar, and the electrostatic fields themselves generate a current density in the interior. A perturbative treatment suggests that these currents in the interior can carry a substantial proportion of the total Hall current.

In a two-dimensional electron system with a strong magnetic field and weak disorder there are two possible mechanisms for the transport of current. For each Landau level and for each edge there is a continuum of edge states which carry the edge current responsible for the Landau diamagnetism. If there is a difference in electrochemical potential between the two edges then the extra electrons occupying one edge provide a net current along the edge proportional to the difference in electrochemical potential. In this case the current is carried by edge states whose energy is close to the electrochemical potential (Fermi energy). On the other hand, in the case of a system with the topology of an annulus or cylinder, an electromotive force can be generated around the system by continuously varying the flux in a solenoid that goes through the center of the annulus or axis of the cylinder [5,6]. In this case the Hall current is directed between the two edges of the system, in a radial direction for the annulus or parallel to the axis for the cylinder, so that all the current must be a bulk current. Such systems have been the subject of a lot of theoretical analysis [4,8–10], but the only experimental realizations I know are by Syphers *et al.* [11] and by Dolgopolov *et al.* [12]. Even in the case of a standard Hall bar it is possible, however, that there is a potential gradient in the uniform region between the two edges, either due to an externally applied electric field, or due in part to electric fields induced by some rearrangement of the electrons in response to the external perturbations, and this potential gradient produces a Hall current density proportional to the local field, so that some of the current is carried in the interior of the system, not at the edges, despite the lack of electron states close to the Fermi energy in this region.

In a strong magnetic field the Landau levels in the interior of a sample are separated by energy intervals (mobility gaps) in which any electronic states generated by random variations of the substrate potential are localized, falling off exponentially from their maximum magnitude over a distance which is the localization length [13–15]. At very low temperatures thermal excitations of holes in occupied Landau levels or electrons in empty Landau levels are negligible, so that the response of a system to a perturbation is determined primarily by these localized states in the neighborhood of the Fermi energy E_F. At the edges of the system the situation is different, as there is a continuum of states extended along the direction of the edge at any energy [6]. In general the current density at a point $\mathbf{r}$ can be written in terms of the Green function as

$$\mathbf{j}(\mathbf{r}) = \lim_{\mathbf{r}'\to\mathbf{r}} \frac{1}{2\pi i} \oint dz \frac{e}{m}(i\hbar\nabla - e\mathbf{A})G(\mathbf{r},\mathbf{r}';z)\,, \quad (1)$$

where the integral is taken around a contour that crosses the real axis at E_F and surrounds the real axis below that value. Here $\mathbf{A}(\mathbf{r})$ is the vector potential, $-e$ is the electron charge, and m is its mass. Because of the exponential localization of states at the Fermi energy (except at the edges of the system), and in the complex plane off the real axis, the Green function in the interior of the

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system depends on small perturbations of the Hamiltonian at a distance from $\mathbf{r}$ only through terms that fall off exponentially with the ratio of the distance to the localization length. Thus, when the Fermi energy lies in a mobility gap in some region of the sample, the current density in that region under static conditions depends only on the conditions in that region, and has only an exponentially small dependence on conditions in distant regions. This statement of the dependence of the current density on local conditions can be combined with Laughlin's [5] argument for the quantum Hall effect to give the result that the Hall current density is proportional to the potential gradient in any region where the Fermi energy is in a mobility gap.

According to the theorem ascribed to Bloch, for which an elegant proof was given many years later by Bohm [16], there is no current circulating around an annulus in its equilibrium state, except for what we now call the "mesoscopic persistent current" [17] that decreases as the circumference of the annulus gets larger. In equilibrium the Landau diamagnetism is manifested by equal and opposite currents on the two edges. In addition, if the potential acting on electrons varies over length scales larger than the magnetic length there will be currents circulating in opposite senses around the maxima and minima of the potential.

Laughlin's argument for the integer quantum Hall effect [5] can be rephrased as a generalization of the argument for Bloch's theorem. Consider an annulus in a strong magnetic field, with the Fermi energy shifted by an amount $-eU$ between the outer edge and the inner edge of the annulus. Both of these Fermi energies lie in the same bulk mobility gap between Landau levels, so the system can remain in a steady state even though it is not in equilibrium. Suppose also that there is a solenoid threading the annulus which can be used to change the vector potential in the annulus without altering the magnetic field. Make an adiabatic change in the flux through the solenoid by one quantum, h/e; the vector potential changes by an amount $\delta\mathbf{A}$ whose integral around the annulus satisfies

$$\oint \delta\mathbf{A}\, d\mathbf{r} = h/e\,. \tag{2}$$

This adiabatic change may shift an integer number N of electrons from one edge to the other, but will leave the system still in a steady state. It can be restored to its original state (apart from some phase twists which change no measurable quantities) by transferring the N electrons back to the one edge from the other. Because the system returns to its original state (apart from unobservable phase factors), the total energy change in this process is zero, and it is made up of the change in the magnetic energy of the current due to the change in vector potential and the change in electrochemical energy of the electrons in transferring them between the two edges at different voltages. This gives the result

$$\delta E = 0 = \int\int \mathbf{j}(\mathbf{r})\delta\mathbf{A}(\mathbf{r})d^2r - NeU = Ih/e - NeU\ , \tag{3}$$

where I is the total current around the annulus. This gives the quantum Hall relation $I = Ne^2U/h$, and, by implication, the Bloch result that the current is zero when the applied voltage U is zero. Mesoscopic currents, which are neglected in this argument, arise from an oscillatory dependence of the current I on the fractional part of the flux threading the annulus.

To show that the electrostatic field in the interior generates a Hall current density, consider now an annulus in which the electrons are confined by a potential which rises at the two edges at R_1 and R_2, and is more or less uniform for r well inside the domain $R_1 < r < R_2$. In this system the current density is zero except in the neighborhood of the two edges, where, from the theorem of Bloch, the two circulating currents must be equal and opposite. This is shown in Fig. 1(a). Next imagine a smooth step of magnitude δV is added to the potential for $r > R_i$, where R_i is some radius in the interior of the

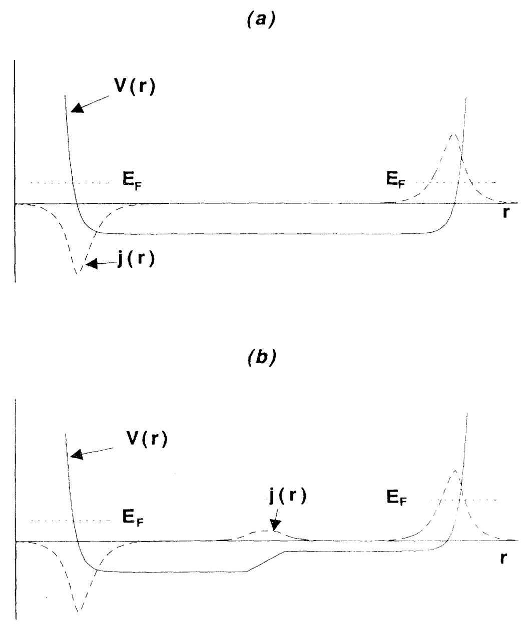

FIG. 1. This shows a potential well $V(r)$ confining the electrons in an annulus, together with the induced current $j(r)$. In (a) the system is in equilibrium, with the same Fermi energy E_F at each edge, and the currents are the equal and opposite currents giving diamagnetism. In (b) a potential step has been introduced in the interior without modifying conditions near the edges, so that the Fermi energy at one edge is raised relative to that at the other by the amount of the potential step. This gives an unbalanced contribution to the current density near the potential step.

annulus, without changing the potential gradient except in the neighborhood of $r = R_i$, and without changing the values of the electron chemical potentials at the two edges relative to the local potential. This is shown in Fig. 1(b). There is now a voltage $\delta V/e$ across the system because of the shift of the electrochemical potential at the outer edge relative to that at the inner edge, so we know from Laughlin's classic argument for the integer quantum Hall effect that the current around the annulus is $Ne\delta V/h$, where N is an integer. We also know from the discussion of Eq. (1) that this extra current must be confined to within a localization length of the step in the potential at R_i. In this way it can be shown that any change in the potential gradient induces a current density normal to the gradient, and proportional to it with a constant of proportionality related to the quantum Hall conductance. This current density does not follow the potential gradient precisely, but is averaged out over a distance of the order of the localization length at the Fermi energy.

Although this argument appears to be restricted to the case of an annulus, it is not actually so restricted. Since the current desity depends only on local conditions, the local relation must be preserved in any geometry. The current density is therefore given by

$$\mathbf{j}(\mathbf{r}) = N\nabla V e^2/h\,, \tag{4}$$

where V is the electrostatic potential and N is an integer, for any region in which there are no mobile states near the Fermi energy.

In an ideal Hall bar at low temperatures current is forced through from one current lead to the other, and voltages are applied at the voltage probes to prevent a net current from flowing through the voltage leads. To a first approximation all the current flows in the edge states, since there are no other mobile states close to the Fermi energy. In such a system there will be an electrostatic field generated by the extra particles in the edge states, and this electrostatic field will produce a distributed Hall current density proportional to the electrostatic field. This is true whether the edge dividing filled and empty states of a Landau level is sharp, as has been considered by Halperin and most subsequent papers, or if it consists of a region of partially occupied states, as Chklovskii, Shklovskii, and Glazman [18] have recently argued. This field will be screened by the mobile electrons in the region beyond the depletion region around the inversion layer. It may also be modified by the carriers that generate the longitudinal resistance, presumably highly activated electrons or holes in the inversion layer.

Similar problems were considered by MacDonald, Rice, and Brinkman [19] and by Heinonen and Taylor [20], and an analytic solution for the effect of a single edge at very low temperatures was derived by Thouless [21]. The boundary conditions considered in Refs. [19,21] are not, however, quite the ones we want to consider here, since it was assumed that a uniform electrostatic field was imposed. The two results that I need from these analyses are that, in the region distant from both edges, the Hall current density is proportional to the electrostatic field, as in Eq. (4), and there is a compression of the electron density by the field gradient, given approximately by

$$\delta\rho = N\nabla^2 V m^*/2hB\,, \tag{5}$$

where m^* is the effective mass of the electrons.

We are not concerned at present with the equilibrium distribution of the charges and diamagnetic currents, but with the changes that occur when a Hall current is imposed on the edges. In the nearly uniform bar, the transfer of electrons from one edge to the other that this implies produces an extra electrostatic field in the interior inversely proportional to the distance from the edge, since the extra (or missing) electrons on each edge form a line charge along the edge. The field is inversely proportional to distance up to distances where screening by carriers in the substrate becomes effective. Equation (4) shows that this electrostatic field gives a current density inversely proportional to the distance from the edge, while Eq. (5) shows that it also leads to a change in density inversely proportional to the square of the distance from the edge.

If we write the extra charge density imposed at the edges as $S(x)$, then the lowest approximation for the resultant potential is

$$V(x) = \int S(x')u(x-x')\,dx'\,, \tag{6}$$

where $u(x)$ is the screened Coulomb potential due to a unit line charge. According to Eq. (5), this change in potential will itself produce a resultant line charge density proportional to its second derivative, and so a set of self-consistent equations may have to be solved.

At large distances, according to Eqs. (4) and (6), the dominant contribution to the current density is

$$j(x) \approx \frac{Ne^2}{h}\int S(x')u'(x-x')\,dx'\,, \tag{7}$$

so that in the absence of external screening the current density falls off from the edge as the inverse of the distance from the edge.

External screening, by charges outside the depletion region, will cut off this long-range contribution at some distance greater than the width of the depletion region. The equations written here are only valid at distances from the edge which are larger than the magnetic length l_0. Therefore the Coulomb law for line charges

$$u'(x) = 1/2\pi\epsilon\epsilon_0 x \tag{8}$$

only holds within the inversion layer for distances $l_{\min} < x < l_{\max}$, where $l_{\min}$ is of the order of l_0 and $l_{\max}$ is of the order of the screening length.

From this formula one can make an estimate of what proportion of the current is carried directly by the electrons added to the edge and what proportion is induced in the interior by the electrostatic field of the added electrons. If we integrate Eq. (7) over positive x we get the induced current in the interior to be

$$I_{\rm int} \approx \delta\rho_{\rm edge}\frac{Ne^3}{2\pi\epsilon\epsilon_0 h}\ln\frac{l_{\rm max}}{l_{\rm min}} , \qquad (9)$$

where $\delta\rho_{\rm edge}$ is the density of added electrons at the edge, and ϵ is the effective dielectric constant.

It is harder to estimate the contribution of the electrons at the edge to the edge currents, since that depends in detail on the form of the confining potential. The maximum current will be obtained if confinement is at a hard wall, as one might get if the inversion layer ends abruptly in a vacuum or an insulating material. In this case a rather crude semiclassical estimate of the mean velocity of edge electrons at the Fermi energy gives the edge current as

$$I_{\rm edge} \approx \delta\rho_{\rm edge} e\sqrt{N\hbar eB}/m^* . \qquad (10)$$

The ratio of these two expressions gives

$$\frac{I_{\rm int}}{I_{\rm edge}} \approx \sqrt{\frac{Ne^3}{2\pi h^3 B}}\frac{m^*}{\epsilon\epsilon_0}\ln\frac{l_{\rm max}}{l_{\rm min}} = \frac{N}{\pi}\sqrt{\frac{1}{2\pi a_B^2\rho}}\ln\frac{l_{\rm max}}{l_{\rm min}} , \qquad (11)$$

where a_B is the Bohr radius for the electron in the semiconductor and ρ is the electron density in the interior. For the usual conditions of operation the coefficient of the logarithm is of the order of 0.2, and the logarithm can be of the order of 5, so the contribution of the distributed currents may be rather large, in fact so large that this perturbative treatment must be improved. If the edge electrons are less tightly confined than I have supposed here the contribution of the edge electrons will be further reduced. Work on this problem is in progress [22].

Fontein *et al.* [23] have used an electro-optic method to measure the potential drop in GaAs under conditions of the quantum Hall effect, and find about 80% of the potential drop at the edges, 20% in the interior. The calculations by Heinonen and Taylor [20] give a much smaller proportion of interior current.

Despite this spreading out of the current density, the Büttiker [7] argument for the importance of the edge voltages is unaffected. It is the electrochemical potential difference between the two edges that determines the magnitude of the Hall current. The distribution in space of the Hall current does not change the arguments that have been presented for this [5–7].

Much of this analysis should be valid also in the regime of the fractional quantum Hall effect. The Green function in Eq. (1) is exponentially localized if the Fermi energy lies in a mobility gap of the quasiparticles, although the localization length is likely to be longer than it is when the Fermi energy lies between Landau levels, so Eq. (4) holds with N replaced by a fraction. There will again be a current response in the interior to the Coulomb field produced by the current-carrying quasiparticle states at the edges.

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