Zero-Temperature Frequency-Dependent Hall Conductivity of the Anderson Insulator

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Using the Kubo formula and a simplified Holstein model to calculate the Hall effect in the Anderson insulator, it is found that at low frequencies $\sigma_{xy} \propto \omega^2$ (with nonanalytical corrections). The coefficient is evaluated for strong localization and found to lead to the same order of magnitude as the usual classical result for the Hall coefficient.

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Recently, Viehweger and Efetov [1] and Zhang, Kivelson, and Lee [2] found that the zero-temperature $\sigma_{xy}(\omega)$ in the Anderson insulator vanishes at low frequencies proportionally to ω^2. This result is rather interesting since the Hall resistivity, which is given by $\rho_{yx}=\sigma_{xy}/(\sigma_{xx}^2+\sigma_{xy}^2)$, is then found to approach a *constant* as $\omega \to 0$. This is due to the leading term of $\sigma_{xx} \sim (i\omega/4\pi)\epsilon_0$ vanishing linearly as $\omega \to 0$. Here ϵ_0 is the dc dielectric constant of the insulator. This observation helps to explain why the small field Hall coefficient appears to change much less than ρ_{xx} near *some* metal-insulator transitions [3,4] as well as around the quantum Hall effect situations [5,6].

Both the above approaches used the linear response, Kubo formula for σ_{xy}, for noninteracting electrons in localized states, averaged over the disorder ensemble to restore the rotational symmetry. Reference [1] used the "supersymmetry" method. The analytical behavior of $\sigma_{xy}(\omega)$ around $\omega=0$ was questioned in Ref. [2], in which a Kramers-Kronig (KK) type relationship was obtained for $\sigma_{xy}/\omega^2 \equiv \pi_{xy}$. The KK integral for $\pi_{xy}(0)$ was proven on general grounds to be convergent, hence π_{xy} was argued to be finite. Our results will support the nonanalytic nature (but, in low orders, just in the imaginary part) of σ_{xy}. The proof of the finiteness of $\pi_{xy}(0)$, however, disregards the (perhaps remote) possibility that the KK integral for $\pi_{xy}(0)$ might actually vanish. Since such a cancellation indeed obviously occurs in the KK integral for, e.g., $\sigma_{xx}(0)$ in the $T=0$ insulator, a calculation of the coefficient of ω^2 in $\sigma_{xy}(\omega)$, which is of interest in its own right, is necessary.

Such a calculation, using a simplification of the Holstein [7,8] model, will be presented below. This is the first step towards addressing the more difficult issue of the critical behavior of σ_{xy} (and ρ_{xy}) on approaching the localization transition from the insulating side when the localization length ξ diverges [9,10]. In the Holstein model a finite σ_{xy} is obtained for a two-site (we shall use the term "site" for any localized state) process by allowing at least the magnetic-field sensitive interference with higher order, e.g., three-site, processes. Our method of calculation differs, however, from that of Holstein [7] and Friedman and Holstein [8] in that we shall use the Kubo formula for $\sigma_{xy}(\omega)$ for the electrons only. The interesting finding is that the terms linear in H in σ_{xy} follow from σ_{xx} upon multiplication by $i\omega H$ (and, of course, by important ξ-dependent factors). Thus the well-known low-frequency behavior of $\sigma_{xx}=i\omega\epsilon_0/4\pi$ becomes

$$\mathrm{Re}\sigma_{xy}=B\omega^2 \tag{1}$$

for the real part, the imaginary part is nonanalytical, B is a constant. A finite σ_{xy} at nonzero ω yields a finite component of the **D** field *perpendicular* to the original polarization of the electromagnetic wave. This is therefore equivalent to the Faraday effect [11]. Indeed, the connection between our picture, based on the Holstein model, and the well-known classical or quantum picture of the Faraday effect will be pointed out towards the end of this Letter.

Before briefly presenting the derivation of our results, we note that the general symmetries of σ_{xy} in fact follow from the Onsager relation $\sigma_{xy}(\omega,H)=\sigma_{yx}(-\omega,-H)$. If rotational symmetry is *restored* by ensemble averaging, it is obvious that $\bar{\sigma}_{xy}(H)=-\bar{\sigma}_{yx}(H)$ (where the bar denotes ensemble averaging). Thus

$$\bar{\sigma}_{xy}(\omega,H)=-\bar{\sigma}_{xy}(-\omega,-H)\,. \tag{2}$$

Therefore the term linear in H must be *even* in ω. Since the ω^0 term is assumed to vanish in the insulator, the lowest order possible term, assuming a power-law form, is $B\omega^2 H$. One has to check that $B\neq 0$ and find its sign [7,8]. Fortunately, the power series expansion holds for the ω^2 term of the real part of Eq. (1).

As in Refs. [1] and [2], we start from the Kubo formula for σ_{xy}. We shall mainly consider the ensemble-averaged quantity, $\bar{\sigma}_{xy}$, without the potentially very important, mesoscopic fluctuation effects. Electron-electron interactions are treated only in the most rudimentary fashion in the self-consistent-field theory:

$$\sigma_{\beta\alpha}=\lim_{\eta\to 0}\sum_{\substack{E_j>\mu \\ E_i<\mu}}\frac{ie^2\omega}{\hbar V}\left(-\frac{\langle i|r_\alpha|j\rangle\langle j|r_\beta|i\rangle}{\omega+\omega_{ji}+i\eta}+\frac{\langle j|r_\alpha|i\rangle\langle i|r_\beta|j\rangle}{\omega-\omega_{ji}+i\eta}\right), \tag{3}$$

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where V is the volume, α and β are Cartesian components, and $\hbar\omega_{ji}=E_j-E_i$, with E_i and E_j single-electron energies, taken respectively below and above the Fermi energy, μ. We note that, whereas only $|\langle j|x|i\rangle|^2$ appears in σ_{xx}, products of matrix elements of x and y appear in σ_{xy}. While the former is positive definite, the latter have arbitrary phases and vanish by symmetry at $H=0$, for $\bar{\sigma}_{xy}$.

The Holstein model [7,8] consists in taking i and j to be localized states on sites 1 (E_1 below μ) and 2 (E_2 above μ), respectively. Transitions between them can occur directly or via (to the next lowest order) an intermediate site, 3, which may be taken, for definiteness, to have an energy $E_3>\mu$. The single-site wave functions $|j\rangle$ ($j=1,\ldots,3$) are taken to be real and orthogonal and to have mutual matrix elements J_{ij}. We take $|J_{ij}|\lesssim|E_i-E_j|$. The effect of tunneling is taken into account by diagonalizing the matrix

$$\mathcal{H}\equiv\begin{pmatrix} E_1 & J_{12} & J_{13} \\ J_{21} & E_2 & J_{23} \\ J_{31} & J_{32} & E_3 \end{pmatrix}, \tag{4}$$

E_i are the unperturbed energies, before switching J_{ij} on. The eigenvectors of $\mathcal{H}$ give the "dressed" site states $|j'\rangle$ and the matrix elements of x and y are evaluated among the latter. We shall work in the Anderson localized regime far enough from the transition so that the localization length ξ is somewhat larger, but of the same order of magnitude as the atomic length, (i.e., the spatial extent of the basis functions $|j\rangle$). For simplicity we may think about the case $E_2-E_1\ll E_3-E_1$, since the transitions are strongest with small energy denominators, although this assumption is easily relaxed. We have in mind the case $|J_{ij}|\sim|J|\ll|E_i-E_j|\sim W$ (although in the end we will let in some cases J approach the order of magnitude of the typical W). To lowest relevant order in $J/|E_i-E_j|$ and neglecting the matrix elements $\langle i|x|j\rangle$ ($i\neq j$), we find

$$\langle 2'|x|1'\rangle=\frac{NJ_{21}}{E_1-E_2}(x_2-x_1)+\frac{J_{23}J_{31}}{(E_1-E_3)(E_2-E_3)}\left(x_3-\frac{x_1+x_2}{2}\right), \tag{5}$$

where N is a normalization constant which may be taken as unity for our purposes and x_j is the average of x in the unperturbed state on the site j, $x_j=\langle j|x|j\rangle$. Higher order terms involving x_2-x_1 were neglected. For a small magnetic field in the z direction and a flux ϕ in the triangle formed by (x_i,y_i), $i=1,2,3$ ($\phi\sim H\xi^2$), the whole effect of H is to change the phase of the product of the J's, $J_{12}J_{23}J_{31}$, by $e^{+i\varphi}$ and that of $J_{13}J_{32}J_{21}$ by $e^{-i\varphi}$ [see (6) below for a definition of φ] which can be regarded as an Aharonov-Bohm (AB) effect. H has to be not so large as to affect the site wave functions, thus we take

$$\varphi\equiv 2\pi\phi e/hc\lesssim 1, \tag{6}$$

where ϕ is equal to H times the oriented area S_{123} of the triangle formed by the points, 1,2,3.

We now obtain the small ω ($\omega\ll\omega_{21}$) behavior of $\mathrm{Re}\sigma_{yx}$ from Eq. (3) by evaluating the imaginary part of the expression in the parenthesis. We note that this contribution is a nonresonant one, the denominator is finite. This is similar (see below) to the evaluation of ϵ_{xx}. We find, denoting $2S_{123}=(x_2-x_1)(y_3-\bar{y})-(y_2-y_1)(x_3-\bar{x})$ with $\bar{x}=(x_1+x_2)/2$, etc.,

$$\mathrm{Re}\sigma_{yx}\cong\frac{e^2}{\hbar V}\frac{S_{123}(J_{12}J_{23}J_{31})^{(0)}\omega^2}{\omega_{21}^3\omega_{13}\omega_{23}}\sin\varphi \tag{7}$$

[where the term even in φ contributes only to the imaginary part of σ_{yx} (i.e., to ϵ_{yx})]. Here $(J_{12}J_{23}J_{32})^{(0)}$ is the value of the product of the J's at $H=0$, whose sign was discussed in Ref. [8]. Note that this has the expected Onsager symmetry in H and ω. To obtain σ_{yx} for the whole sample we have to sum over all 123 groups. Consider, for example, the term linear in H. Since, by (6) φ too is proportional to S_{123} we find for the contribution of each (123) group:

$$\mathrm{Re}\bar{\sigma}_{yx}=\frac{\pi e^2}{\hbar V}(S_{123})^2\frac{J^3}{(\hbar\omega_{21})^3}\frac{\omega^2}{\omega_{13}\omega_{23}}H/\phi_0 \sim\frac{e^3\xi^4}{\hbar^2cV}\left(\frac{\omega}{\omega_\xi}\right)^2 H. \tag{8}$$

Here the occurrence of S_{123}^2 guarantees a nonzero ensemble average of σ_{yx}. The order of magnitude estimate in the last term is valid for strong localization, $\xi\sim a$. Here ω_ξ is the level separation on scale ξ $\{\hbar\omega_\xi=[\xi^d n(0)]^{-1}\}$, $J/\hbar\omega_\xi$ was taken to be of order unity. We note the similarity to the calculation of ϵ_{xx} (see, e.g., Ref. [9]). In the strong localization case we find for ρ_{xy}, the number of (123) groups being V/ξ^3 for 3D, and using [9] $\sigma_{xx}\sim i\omega(\xi/\lambda)^2$ for $\xi\sim a$, λ being the screening length,

$$\rho_{xy}\sim\frac{a^3}{ec}H, \tag{9}$$

where a is the microscopic length. Thus, in the full quantum calculation the classical expression for $R_H=-1/nec$ is recovered in order of magnitude for strong localization. This is rather nontrivial, since the effect of the magnetic field in this calculation is purely through the AB phase. It is remarkable that the result is almost *as if* the classical Lorentz force prevailed for the electron with average drift along the x direction. The sign of such a result was discussed in Ref. [8] and will be considered below.

It is less straightforward to perform this calculation, as was done [9] for ϵ_{xx}, approaching the localization transition, $\xi/a\to\infty$. This requires a careful evaluation of the matrix elements $\langle j|x|i\rangle$, etc. and their field dependences. Since ρ_{xy} is of the same order of magnitude deep in the metallic and insulating ranges, one might heuristically expect it to stay the same throughout the transition. On the other hand, naively using our result for large ξ without

justification and disregarding local-field corrections lead to a divergence [10] of σ_{xy} when $\xi \to \infty$. Another nontrivial issue is the evaluation of $\sigma_{xy}(T)$, even at $\omega \to 0$, at finite temperatures. It might be expected [12] that the $T=0$, finite ω results would be valid down to a cutoff frequency ω_0 which depends on temperature and the usual electron-phonon mechanisms, etc. Assuming that $\sigma_{\alpha\beta}(T) \sim \sigma_{\alpha\beta}(T=0, \omega \sim \omega_0(T))$ will yield the temperature independence of $\rho_{xy}(T)$ for strong localization. This obviously needs further treatment and may depend on the type of insulator considered [2].

As mentioned above, having a $\sigma_{xy}(\omega) \neq 0$ leads to a rotation of the polarization direction of an electromagnetic wave with frequency ω at $H \neq 0$. This is the Faraday effect. Indeed, our calculation can be simply viewed as a specific quantum model for the usual classical picture of the latter [11], again based on the Lorentz force perpendicular to the original electric field direction. In fact, the sign of our result [Eq. (8)] also agrees with that of the classical Faraday effect. For example, a negative J, appropriate to electrons in an ordinary conduction band, yields a Hall effect corresponding to the classical result for electrons. Measurements of the Faraday effect in the Anderson insulator, especially near the localization transition, are suggested.

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