Analytical Solution of the Off-Equilibrium Dynamics of a Long-Range Spin-Glass Model

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We study the nonequilibrium relaxation of the spherical spin-glass model with p-spin interactions in the $N \to \infty$ limit. We analytically solve the asymptotics of the magnetization and the correlation and response functions for long but finite times. Even in the thermodynamic limit the system exhibits "weak" (as well as "true") ergodicity breaking and aging effects. We determine a functional Parisi-like order parameter $P_d(q)$ which plays a similar role for the dynamics to that played by the usual function for the statics.

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Spin-glass dynamics has been a subject of continuous interest in the past years. Experimentally, spin-glass dynamics below the critical temperature is characterized by aging effects and very slow relaxations [1]. In long-range mean-field models one knows that the phase space is broken into ergodic components [2]. Sompolinsky [3] described a dynamics for these models allowing for barrier penetration in very long times (diverging as $N \to \infty$).

In realistic systems, on the one hand mean field is not exact and on the other hand one cannot perform an experiment in infinite times, and one actually sees at most "weak" ergodicity breaking.

Recently, Bouchaud has proposed a phenomenological scenario with both "true" and weak ergodicity breaking [4]. The question then arises as to if and how simple long-range microscopic systems (for which mean field is exact) can model these phenomena. To the best of our knowledge, an analytic description is lacking.

The main purpose of this paper is to show, in a very simple mean-field model, the asymptotics of which we solve analytically that this is indeed so; in the thermodynamic limit "true" and weak ergodicity breaking coexist, and in a sense are complementary. To this end we solve the dynamics of the p-spin spherical model ($p > 2$) first introduced in Ref. [5], *setting $N \to \infty$ from the outset*, starting from a given configuration, for long (but not diverging with N) times. Remarkably, such a simple model exhibits a very similar behavior to that of realistic models (e.g., 3D Edward-Anderson) in the autocorrelation function for long times [6] and in simulations of aging experiments [7].

It should be stressed that we are discussing a different physical situation from the Sompolinsky dynamics, which was analyzed in Ref. [8]. We do not have here any time scale dependent on N or any other "regularization" parameter. Surprisingly, one can establish *formal* contact with Sompolinsky's equations by defining a variable $\mathcal{T} = \ln(t'/t)$ which plays the role of the "time" there. This will be further explained in a separate work in the context of the Sherrington-Kirkpatrick (SK) model [9].

The spherical p-spin glass model is defined by the Hamiltonian

$$H = -\sum_{i_1<\cdots<i_p}^{N} J_{i_1\cdots i_p} s_{i_1}\cdots s_{i_p} + \frac{1}{N^{r-1}} \sum_{i_1<\cdots<i_r}^{N} h_{i_1\cdots i_r} s_{i_1}\cdots s_{i_r} \; . \tag{1}$$

The spin variables verify the spherical constraint $\sum_{i=1}^{N} s_i^2(t) = N$. The interaction strengths are independent random variables with a Gaussian distribution with zero mean and variance $\overline{(J_{i_1\cdots i_p})^2} = p!/2N^{p-1}$. The overbar stands for the average over the couplings. Additional source terms ($h_{i_1\cdots i_r}$ time independent) have been included; if $r = 1$ the usual coupling to a magnetic field h_i is recovered.

The relaxational dynamics is given by the Langevin equation

$$\Gamma_0^{-1}\, \partial_t s_i(t) = -\beta \frac{\delta H}{\delta s_i(t)} - z(t) s_i(t) + \xi_i(t) \; . \tag{2}$$

Γ_0 determines the time scale and will be henceforth set to 1. The second term on the right-hand side enforces the spherical constraint while $\xi_i(t)$ is a Gaussian white noise with zero mean and variance 2. The mean over the thermal noise is hereafter represented by $\langle \cdots \rangle$. As will be shown below, the dynamical equations plus the spherical constraint impose $z(t) = 1 - p\beta\mathcal{E}(t)$ with $\mathcal{E}(t)$ the energy per spin. We choose as initial configuration $s_i(0) = 1\ \forall i$, though any other choice is equivalent.

The mean-field sample-averaged dynamics for $N \to \infty$ is entirely described by the evolution of the two-time correlation and the linear response functions,

$$C(t,t') \equiv \frac{1}{N} \sum_{i=1}^{N} \overline{\langle s_i(t) s_i(t') \rangle},$$

$$G(t,t') \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \overline{\langle s_i(t) \rangle}}{\partial h_i(t')} \; .$$

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The dynamical equations for them can be obtained from Eq. (2) through standard functional methods (see, e.g., Ref. [10]),

$$\frac{\partial C(t,t')}{\partial t} = -\left[1-p\beta\,\mathcal{E}(t)\right] C(t,t') + 2\,G(t',t) + \mu\int_0^{t'} dt''\, C^{p-1}(t,t'')\, G(t',t'') + \mu\,(p-1)\int_0^{t} dt''\, G(t,t'')\, C^{p-2}(t,t'')\, C(t'',t')\ , \tag{3}$$

$$\frac{\partial G(t,t')}{\partial t} = -[1-p\beta\,\mathcal{E}(t)]\, G(t,t') + \delta(t-t') + \mu\,(p-1)\int_{t'}^{t} dt''\, G(t,t'')\, C^{p-2}(t,t'')\, G(t'',t')\ , \tag{4}$$

with $\mu \equiv p\beta^2/2$. These equations hold for all times t and t'. At equal times $C(t,t)=1$, $\lim_{t'\to t^-} G(t,t')=1$ and $\lim_{t'\to t^\pm} \partial_t C(t,t') = \pm 1$. $\mathcal{E}(t)$ can be identified as the energy per spin multiplying Eq. (2) by $s_i(t')$, averaging over the noise and the couplings and taking the limit $t'\to t$. Furthermore, with the definition

$$I^r(t) \equiv \frac{r}{N^r}\sum_{i_1<\cdots<i_r} \left.\frac{\partial\overline{\langle s_{i_1}(t)\cdots s_{i_r}(t)\rangle}}{\partial h_{i_1\cdots i_r}}\right|_{h=0} = r\int_0^t dt''\, C^{r-1}(t,t'')\, G(t,t''), \tag{5}$$

Eq. (3) implies $\mathcal{E}(t) = -(\beta/2) I^p(t)$.

Since we are interested in the nonequilibrium dynamics, we solve these equations with the only assumption of causality. We take $t>t'$ for definiteness and we focus on the *low* temperature phase. The system (3) and (4) can be solved numerically step by step in a manner reminiscent of Ref. [11]. The numerical solution suggests the following scenario for the asymptotic regime $t>>1$ which we later confirm analytically. The time axis t' is divided into three distinct zones with different behaviors.

(i) For $\tau\equiv t-t'$ finite, $\tau/t\to 0$ asymptotically, time homogeneity and the fluctuation-dissipation theorem (FDT) hold; i.e., $G_{\rm FDT}(\tau) = -\Theta(\tau)\,\partial_\tau C_{\rm FDT}(\tau)$. For large values of τ (but still small compared to t) $C_{\rm FDT}(\tau)$ tends to a value q and $G_{\rm FDT}(\tau)$ tends to zero.

(ii) If t' is such that $(t-t')/t \sim O(1)$, the relevant (adimensional) independent variable turns out to be $\lambda \equiv t'/t$ $(0<\lambda<1)$. In this sector the correlation and rescaled response functions depend on λ as $C(t,t') = q\,\mathcal{C}(\lambda)$ and $t\,G(t,t') = \mathcal{G}(\lambda)$. Since q is the limiting value of $C(t,t')$ in the previous regime, $\mathcal{C}(1)=1$.

(iii) Finite times t' correspond to $\lambda=0$ in rescaled variables. In particular, for $t'=0$ we have the magnetization $m(t) = C(t,0)$.

We now proceed to solve the resulting equations within this asymptotic scenario.

(i) If t' is such that the system is in the FDT regime Eq. (3) yields

$$\left(\frac{\partial}{\partial\tau}+1\right) C_{\rm FDT}(\tau) + (\mu + p\beta\,\mathcal{E}_\infty)\,[1-C_{\rm FDT}(\tau)] = \mu\int_0^\tau d\tau''\, C_{\rm FDT}^{p-1}(\tau-\tau'')\,\frac{dC_{\rm FDT}}{d\tau''}(\tau'')$$

with the asymptotic energy $\mathcal{E}_\infty$ (N.B. "∞" is understood as a limit taken *after* $N\to\infty$) given by

$$\mathcal{E}_\infty = -\frac{\beta}{2}\left[(1-q^p) + pq^{p-1}\int_0^1 d\lambda''\,\mathcal{G}(\lambda'')\mathcal{C}^{p-1}(\lambda'')\right]\ . \tag{6}$$

The correlation decays to a value q determined by

$$1 - p\beta\,\mathcal{E}_\infty + \mu\,(1-q^{p-1}) = -\frac{1}{1-q}\ . \tag{7}$$

This equation appears in the dynamics *à la* Sompolinsky of this model [8]. The solution for q as well as the decay law require solving the coupled system (6) and (7) which involves the previous history through the λ integration.

(ii) The dynamic equations for the regime $0<t'/t<1$ reduce to two coupled equations for $\mathcal{C}(\lambda)$ and $\mathcal{G}(\lambda)$ in which, consistently, all times enter *only* through λ:

$$0 = \mathcal{G}(\lambda)\left[-(1-q)^{-1} + \mu\,(1-q)(p-1)\,q^{p-2}\,\mathcal{C}^{p-2}(\lambda)\right] + \mu\,(p-1)\,q^{p-2}\int_\lambda^1 \frac{d\lambda''}{\lambda''}\,\mathcal{G}(\lambda'')\,\mathcal{C}^{p-2}(\lambda'')\,\mathcal{G}\left(\frac{\lambda}{\lambda''}\right)\ , \tag{8}$$

$$0 = \mathcal{C}(\lambda)\left[-(1-q)^{-1} + \mu\,(1-q)\,q^{p-2}\,\mathcal{C}^{p-2}(\lambda)\right] + \mu\,q^{p-2}\int_0^\lambda \frac{d\lambda''}{\lambda}\,\mathcal{C}^{p-1}(\lambda'')\,\mathcal{G}\left(\frac{\lambda''}{\lambda}\right) + \mu\,(p-1)\,q^{p-2}\int_0^1 d\lambda''\,\mathcal{G}(\lambda'')\,\mathcal{C}^{p-2}(\lambda'')\,\mathcal{C}\left[\left(\frac{\lambda}{\lambda''}\right)^{{\rm sgn}(\lambda''-\lambda)}\right]\ . \tag{9}$$

Equation (8) in $\lambda=1$ admits the solution $\mathcal{G}(1)=0$ which implies $\mathcal{G}(\lambda)\equiv 0$ and this is the *high* temperature asymptotics. In the *low* temperature phase a nontrivial $\mathcal{G}(\lambda)$ is possible provided the first square bracket in (8) evaluated in $\lambda=1$ is zero; this fixes the value q. From Eq. (8) it also follows $\mathcal{G}(1) = x\,q\,\mathcal{C}'(1)$ (the prime denotes derivative with respect to λ) with $x\equiv(p-2)(1-q)/q$. It is now easy to see that the system (8) and (9) with $\mathcal{G}(\lambda) = x\,q\,\mathcal{C}'(\lambda)$ simplifies to a single equation. With this ansatz the system of equations has the family of (exact) solutions

$$\mathcal{C}(\lambda) = \lambda^\gamma \Longleftrightarrow C(t,t') = q\left(\frac{t'}{t}\right)^\gamma . \tag{10}$$

One can show that all asymptotic solutions are related by a local symmetry in the asymptotic equations [12]. Nevertheless, solutions (10) are the only ones having a nonconstant asymptotic limit of $C(t, \lambda t)$ $(0 < \lambda < 1)$ as $t \to \infty$.

In order to determine γ a careful matching between this solution and the ones associated with other sectors has to be made. We do not do this here analytically. Numerical solutions to the system (3) and (4) show that γ increases with T and decreases with p. For instance, $\gamma \simeq 0.25$ for $p = 5$ and $T = 0.2$.

The solution found for the regime (ii) allows us to calculate all the integrals (5) in the large t limit

$$I_\infty^r = 1 - q^r(1-x) . \tag{11}$$

In particular,

$$\mathcal{E}_\infty = -\frac{\beta}{2} I_\infty^p = -\frac{\beta}{2}\left[1 - q^p\left(1 - \frac{(p-2)(1-q)}{q}\right)\right] . \tag{12}$$

We now have everything that is required to solve the FDT relaxation, Eqs. (6) and (7), which *for this value of* $\mathcal{E}_\infty$ imply a power law decay of correlations [8] to a value q given by

$$\frac{1}{p-1} = \mu\, q^{p-2}\,(1-q)^2 . \tag{13}$$

Interestingly enough, the expressions just derived for the energy $\mathcal{E}_\infty$ and q have a particular significance within the Thouless-Anderson-Palmer (TAP) formalism, as we shall see below.

(iii) Finally, we consider the finite t' regime. We already know that for large t correlations relax to zero; we now study the asymptotics. Inserting the behavior $C(t,t') \sim t^{-\alpha} c(t')$ in Eqs. (3) and (4) and using the previous results, we find $\alpha = \gamma$; i.e., the exponent for t is the same as in the previous regime.

The dynamic transition takes place at the temperature at which $x = 1$ in a manner that resembles the static transition [8, 13]; at that point the threshold energy coincides with the paramagnetic energy.

The numerical solutions show that the asymptotic regime is well established already for (adimensional) times $t \sim 100$.

Let us now discuss in more detail some of the implications of this solution. For *any* waiting time t_w there exists a sufficiently large τ such that $C(\tau + t_w, t_w)$ tends to zero. After a (large) t_w, the correlation function has a short transient after which it goes as $q\,(t_w/\tau + t_w)^\gamma$; hence its decay rate is inversely proportional to t_w (an aging effect). We also have that for t large, the magnetization falls to zero as $t^{-\gamma}$.

In addition to a strong short-term memory, the system possesses a weak, long-term memory, which is responsible for the observable aging effects in the susceptibility. Consider, for example, the following procedure: we let the system evolve in the absence of field during a large time t_w at which we apply a small field during a further time t. Using the previous results a short calculation shows that the plot $m(\tau + t_w)$ vs $\ln\tau$ has an inflection point at $\tau = (1/\gamma)t_w$. This is very much like the zero field cooled procedure though we are starting here from a fully magnetized initial state [1, 7].

We expect other models, such as the Potts glass (for more than three components) and the p-spin Ising model (for not too low temperatures), to have a similar dynamics to the one presented here. The SK model instead has a more complicated behavior [9].

Let us now turn to the physical interpretation of this solution. We start by briefly describing the structure of the TAP free-energy landscape in terms of the magnetizations m_i and $q \equiv (1/N)\sum_i^N m_i^2$ [13].

The paramagnetic minimum $q = 0$ exists for all temperatures. At fixed temperature T, below a certain temperature T^*, the TAP equations admit many other solutions with $q > 0$. These solutions have free energies between $f_{\min}(T)$ and an upper threshold $f_{\rm th}(T)$. The value of q of each solution is larger the lower the free energy, attaining its smallest value $q_{\rm th}(T)$ at the threshold. Moreover, the ordering of the solutions in free energy does not change with temperature; i.e., there is no "chaoticity" with respect to temperature, and they neither merge nor coalesce.

Using standard methods [14] one can calculate the typical spectrum of the free-energy Hessian in a local minimum, to find that it corresponds to a "shifted" semicircle law. The lowest eigenvalue $\lambda_{\min}$ is greater than zero for free energies below the threshold. Near the threshold $\lambda_{\min}$ drops to zero as

$$\lambda_{\min}(f,T) \sim \frac{p}{q_{\rm th}}(f_{\rm th} - f),$$

where f is the free energy of the solution considered. This is unlike the SK model for which *all* minima are expected to have $\lambda_{\min} = 0$ [14].

For subthreshold free energies, the potential being smooth and $\lambda_{\min} > 0$, we have well-defined minima with no "zero modes," separated by $O(N)$ barriers. In particular, this was shown within the replica approach for the lowest minima that dominate the Gibbs measure [13]. Minima with free energies in a small neighborhood of $f_{\rm th}$ are separated by barriers of $O(1)$ of all magnitudes.

A direct evaluation of the energy and q at the threshold [13] gives for all temperatures $(T < T^*)$ precisely expressions (12) and (13). Hence we have found that for low temperatures the off-equilibrium dynamics is dominated by this threshold level. This is to be contrasted with the equilibrium (Gibbs measure) situation in which lower lying (subthreshold) states dominate [13]. Since subthreshold states have $\lambda_{\min} > 0$, exponential decays would be expected within them, in contrast with the results found here.

We have hence the following picture for the *low* temperature dynamics. The system first relaxes towards the threshold level. As discussed in the TAP approach, the trapping times in subthreshold states are expected to diverge with N. However, the system does not go below the threshold: shortly above this level it remains slowly touring an extended region of $O(1)$ ruggedness with $f \simeq f_{\rm th}$.

This is seen from the behavior of $C(\tau + t_w, t_w)$ for large t_w: at relatively small τ the system relaxes in a way that does not depend on t_w to a value q which one can picture as the "width" of the channels. The system, however, remains in *any* spatial neighborhood of size q a small time compared with t_w. For larger τ $[O(t_w)]$ it eventually drifts away, signaling the fact that no *bona fide* states of size q are actually visited. However, the trapping in a region of size q tends to be more stable for increasing t_w, as reflected by the fact that the decay rate of the correlation function beyond q for large τ is inversely proportional to t_w.

In finite N simulations one can expect a similar scenario with the only difference being that the system is now able to penetrate a certain (N-dependent) extent below the threshold. Indeed, Monte Carlo simulations with $N = 256$ support this picture [15].

In order to better characterize this dynamic process, we consider the generating function $P_d(q)$ of the generalized susceptibilities:

$$\lim_{t\to\infty} \lim_{N\to\infty} \left[1 - \frac{r}{N^r} \sum_{i_1 < \cdots < i_r} \frac{\partial \overline{\langle s_{i_1}(t) \cdots s_{i_r}(t) \rangle}}{\partial h_{i_1 \cdots i_r}} \bigg|_{h=0} \right] = \int_0^1 dq'\, P_d(q')\, q'^r \ . \quad (14)$$

If in this equation we take the reverse order of limits we obtain the usual Parisi function $P(q)$ [2]. In our case $P_d(q)$ is given by [cf. Eqs.(5) and (11)]

$$P_d(q') = x\,\delta(q') + (1-x)\,\delta(q'-q) \ . \quad (15)$$

Note that in this model $P_d(q) \neq P(q)$.

In the low temperature phase, even if $C(\tau + t_w, t_w) \to 0$, $P(q) \neq \delta(q)$, as happens in the high temperature phase. This excludes the interpretation of the low temperature dynamics as an equilibration in a state with $q = 0$.

A probabilistic interpretation of $P_d(q)$ on the lines of the static $P(q)$ is immediate if one can assume that the system achieves a Boltzmann distribution in a restricted sector of phase space at each (large) time, i.e., the Kramers approximation [16], the validity of which remains to be proved in this context.

It may come as a surprise that analytic results can be obtained at all in such nonequilibrium situations: the underlying reason is the weakness of the long-term memory of the system. We expect that whenever this is the case, analytical results could be obtained.

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