Relativistic Disks as Sources of the Kerr Metric

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Many spacetimes are known which should represent gravitational fields outside rotating sources, but for which no satisfactory sources have been found. We indicate that these spacetimes can arise as the metrics of counterrotating thin disks of finite mass, consisting of two streams of collisionless particles, circulating in opposite directions with different velocities. We show how such disks can act as exact sources of all types of the Kerr metric. In their central regions they can produce ergoregions, the velocities can approach that of light, and the redshifts can become arbitrarily large.

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Although much effort has been devoted to discovering exact solutions of Einstein's equations since the birth of general relativity, there are only a few solutions which are considered as "physically acceptable." New generating techniques have been developed to produce asymptotically flat stationary axisymmetric vacuum spacetimes, but vacuum solutions are burdened by singularities which are difficult to handle. What is lacking are *physical sources* which would produce the vacuum metrics.

A specific role is played by the Kerr metric [1] (the discovery of which, thirty years ago, has been considered [2] to be "among the most important astronomical discoveries of our time . . ., the only discovery in astronomy comparable to the discovery of an elementary particle in physics"). Thanks to the black-hole uniqueness theorems, the Kerr metric represents the unique solution describing all rotating vacuum black holes. Nevertheless, although the cosmic censorship conjecture, on which the physical relevance of the Kerr metric rests, is a very plausible hypothesis (cf., e.g. [3]), it remains one of the central unresolved issues in relativity. It would thus support the significance of the Kerr metric if a physical source were found which produces the Kerr field. The situation would then resemble the case of the spherically symmetric Schwarzschild metric which can represent both a black hole and the external field due to matter.

This, of course, has been realized by many workers. The review on the "Sources for the Kerr Metric" [4], written in 1978, contains 71 references, and concludes with, "Destructive statements denying the existence of a material source for the Kerr metric should be rejected until (if ever) they are reasonably justified." The most recent work we are aware of appeared in 1991, and gives "a toroidal source," consisting of "a toroidal shell. . ., a disk. . . and an annulus of matter interior to the torus" [5]. The masses of the disk and annulus are negative. To summarize in Hermann Bondi's way, the sources suggested so far for the Kerr metric are not the easiest materials to buy in the shops. . . .

The situation is somewhat different in the special case of the *extreme* Kerr metric, where there is a definite relationship between mass and angular momentum. The numerical study [6] of uniformly rotating disks indicated how the extreme Kerr geometry forms around the disk in the "ultrarelativistic" limit. These numerical results were recently supported by the important analytical work [7]. However, in the case of a general Kerr metric no physical sources have been found so far.

Very recently Bičák, Lynden-Bell, and Katz [8] (BLK in the following) have shown that most vacuum *static* Weyl solutions can arise as the metrics of counterrotating relativistic disks (see [8] for other references on relativistic disks). The simple idea which inspired the work of BLK is commonly used in Newtonian galactic dynamics [9]: Imagine a point mass placed at a distance b below the center $\rho=0$ of a plane $z=0$. This gives a solution of Laplace's equation above the plane. Then consider the potential obtained by reflecting this $z\geq 0$ potential in $z=0$ so that a symmetrical solution both above and below the plane is obtained. It is continuous but has a discontinuous normal derivative on $z=0$, the jump in which gives a positive surface density on the plane. In galactic dynamics one considers general line distributions of mass along the negative z axis and, employing the device described above, one finds the potential-density pairs for general axially symmetric disks. In fact, Bičák, Lynden-Bell, and Pichon [10] found an infinite number of new static solutions of Einstein's equations starting from realistic potentials used to describe flat galaxies, as given recently by Evans and de Zeeuw [11].

In this Letter we wish to demonstrate that a similar method works also for axisymmetric, reflection symmetric, and *stationary* spacetimes. It is important to realize that although now no metric function solves Laplace's equation as in the static case, we may view the procedure described above as the *identification* of the hypersurface $z=b$ with $z=-b$. The field then remains continuous but the jump of its normal derivative induces a matter distribution in the disk which arises due to the identification. What remains to be seen is whether the material can be "bought in the shops."

This idea can be employed for all known asymptotically flat stationary vacuum spacetimes, e.g., for the Tomi-

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matsu-Sato solutions, for the "rotating" Curzon solution, or for other metrics (cf. [12] for references). Here we shall illustrate the procedure for the simplest, but most interesting case—the Kerr metric.

Any stationary axisymmetric vacuum metric can be written in canonical coordinates (t,φ,ρ,z) in the form [12]

$$ds^2=f^{-1}[g(d\rho^2+dz^2)+\rho^2 d\varphi^2]-f(dt+Ad\varphi)^2\,, \tag{1}$$

where f,g,A are functions of ρ,z (we put $c=G=1$). Spheroidal coordinates (x,y) are commonly used, and we introduce both prolate ($\kappa=+1$) and oblate ($\kappa=-1$) ones since we wish to include all types of Kerr metrics:

$$\begin{aligned} &z=\lambda xy,\quad \rho=\lambda[(x^2-\kappa)(1-y^2)]^{1/2}\,,\\ &\lambda=\text{const}\,. \end{aligned} \tag{2}$$

For the Kerr solution (mass M, specific angular momentum $a\geq 0$) the functions in (1) are ratios of polynomials [12]:

$$\begin{aligned} &f=L/E,\quad g=L/F,\quad A=B/L\,,\\ &L=p^2x^2+q^2y^2-1,\quad E=(px+1)^2+q^2y^2\,,\\ &F=p^2(x^2-\kappa y^2),\quad B=2Mq(1-y^2)(px+1)\,,\\ &\kappa p^2+q^2=1,\quad q=a/M,\quad \lambda=Mp\,. \end{aligned} \tag{3}$$

Here $\kappa=1$ for $a<M$ and $\kappa=-1$ for $a>M$. If $a=M$, then $\kappa=0$, $\lambda=M$, and $p=q=1$.

Now we identify the "planes" $z=b=\text{const}>0$ and $z=-b$ which will lead to disks with zero radial pressure. With the Kerr geometry the matching is more complicated than in the static cases and, therefore, we turn to Israel's covariant formalism (see [13] for its recent exposition). This enables us, using Einstein's field equations, to link the surface stress-energy tensor $S_{(a)(b)}$ of the disk arising from this identification to the jump $[K_{(a)(b)}]=K_{(a)(b)}|_{z=+b}-K_{(a)(b)}|_{z=-b}$ of normal extrinsic curvature across the timelike hypersurface Σ given by $z=b$ (or $z=-b$). The tetrad indices are denoted by $(a),(b),\ldots$, with the tetrad vectors being chosen so that three vectors tangent to Σ are just $e^{\mu}_{(a)}=\delta^{\mu}_{(a)}$ ($\mu=0,1,2,3,\{x^{\mu}\}=\{t,\phi,\rho,z\},a=0,1,2$), while $n^{\mu}=(0,0,0,1/\sqrt{g_{zz}})$ is the unit normal, as required [13].

As a consequence of Einstein's field equations we find the nonvanishing components of the surface stress-energy tensor to be

$$\begin{aligned} &S_{(0)(0)}=\frac{ZLF}{E^2}\left[\frac{L'}{L}+\frac{F'}{F}-2\frac{E'}{E}\right],\\ &S_{(0)(1)}=\frac{ZBF}{E^2}\left[\frac{B'}{B}+\frac{F'}{F}-2\frac{E'}{E}\right],\\ &S_{(1)(1)}=\frac{ZF}{L}\left[\rho_1^2\left[\frac{L'}{L}-\frac{F'}{F}\right]+2\frac{B^2}{E^2}\left[\frac{B'}{B}-\frac{E'}{E}\right]\right], \end{aligned} \tag{4}$$

where L,E,F,B are given by (3), $\rho_1^2=\rho^2-B^2/E^2$,

$$Z=\frac{1}{16\pi}\left[\frac{E}{F}\right]^{1/2}\frac{2y_+}{\lambda(x^2-\kappa y^2)},\quad y_+=y|_{z=+b}\,, \tag{5}$$

and f' ($f=L,F,E$, etc.) are defined by the relation $[f_{,z}]^{z=b}_{z=-b}=2y_+f'/\lambda(x^2-\kappa y^2)$. For the Kerr metric

$$\begin{aligned} &L'=2x(p^2x^2-q^2y^2+2q^2-1)\,,\\ &E'=2p(x^2-\kappa)(px+1)+2xq^2(1-y^2)\,,\\ &F'=2x[p^2x^2+(1-q^2)(y^2-2)]\,,\\ &B'=2M(q/p)(1-y^2)[q^2-(px+1)^2]\,. \end{aligned} \tag{6}$$

[Above in (5) and (6), both x and y are evaluated at $z=b$; by inverting (2) one finds $x|_{z=+b}=x|_{z=-b}$, $y^2|_{z=+b}=y^2|_{z=-b}$. The formulas below are valid for all values of κ, unless it is explicity stated that, e.g., $a>M$, i.e., $\kappa=-1$.] Equations (4)-(6) give the stress-energy tensor of the disks.

Let us now show that the disks may be interpreted as being made of two streams of collisionless particles that circulate in opposite directions. In order to see this, we find, at each radius ρ and for $z=b$, the preferred observer for whom the stress-energy tensor (4) acquires a diagonal form. Let his 4-velocity (the timelike eigenvector of the tensor) read

$$V^{\mu}=N(1,\Omega,0,0)\,, \tag{7}$$

and his unit spatial base vector in the ϕ direction be

$$W^{\mu}=J(\beta,1,0,0)\,. \tag{8}$$

The conditions $V^{\mu}V_{\mu}=-1$, $W^{\mu}W_{\mu}=1$, and $V^{\mu}W_{\mu}=0$ determine three of the parameters entering (7) and (8) in terms of the fourth, of Ω say. Assume now a tensor $T^{\mu\nu}$ [which will be calculated from (4)-(6)] has nonvanishing components T^{00}, T^{01}, T^{11}. Then, by choosing $\Omega=\{(T^1_1-T^0_0)-[(T^1_1-T^0_0)^2+4T^0_1T^1_0]^{1/2}\}/2T^0_1$, $T^{\mu\nu}$ can be cast into the form

$$T^{\mu\nu}=\sigma V^{\mu}V^{\nu}+PW^{\mu}W^{\nu}\,, \tag{9}$$

where

$$\sigma=\frac{T^{00}-\beta^2T^{11}}{N^2(1-\beta^2\Omega^2)},\quad P=\frac{T^{11}-\Omega^2T^{00}}{J^2(1-\beta^2\Omega^2)}\,. \tag{10}$$

Hence, the observers circulating with the 4-velocity (7) will see the diagonal form (in t, ϕ components) of $T^{\mu\nu}$ with

$$\begin{aligned} &T^*_{(0)(0)}=T^{\mu\nu}V_{\mu}V_{\nu}=\sigma,\quad T^*_{(0)(1)}=T^{\mu\nu}V_{\mu}W_{\nu}=0\,,\\ &T^*_{(1)(1)}=T^{\mu\nu}W_{\mu}W_{\nu}=P\,. \end{aligned} \tag{11}$$

We call such observers "ϕ isotropic" (FIOs) since the isotropy concerns the ϕ direction only.

If $\sigma\geq P>0$, FIOs can consider the stress-energy tensor (11) as representing two equal streams of collisionless particles that circulate in opposite directions with the *same* velocity

$$U^* = (P/\sigma)^{1/2}. \tag{12}$$

If σ_p is the surface proper rest mass density of one stream (measured in axes that move with it), then the surface density of its rest mass measured by an FIO is $\frac{1}{2}\sigma_0 = \sigma_p[1-(U^*)^2]^{-1/2}$. The surface energy density of the *pair* of streams is

$$T^*_{(0)(0)} = \sigma = \sigma_0/[1-(U^*)^2]^{1/2} = 2\sigma_p/[1-(U^*)^2]. \tag{13}$$

The tangential pressure caused by the counter rotation is $T^*_{(1)(1)} = \sigma(U^*)^2$. The sum of the proper rest mass surface densities of both streams is simply

$$2\sigma_p = \sigma - P. \tag{14}$$

The condition that the velocity of the streams U^* does not exceed the velocity of light is just the dominant energy condition (see, e.g., [3]).

What is the relation of FIOs to the locally nonrotating frames (LNRFs)? FIOs *rotate* with respect to LNRFs with the velocity given by

$$V = [g_{11}/(-g_{00}g_{11}+g_{01}^2)^{1/2}](\Omega-\omega), \tag{15}$$

where $\omega = -g_{\varphi t}/g_{\varphi\varphi} = A/(\rho^2 f^{-2} - A^2)$. Therefore, the streams circulate with *different* velocities in LNRFs and, of course, with respect to "distant stars." That is why the disks produce *stationary* rather than static fields.

The physical quantities introduced in (12)-(15) are given in terms of the metric (1)-(3), and the stress-energy tensor (4)-(6). Hence, *all the physical quantities describing the disks are given analytically.* The resulting expressions, however, are so complicated that it is only reasonable to exhibit them graphically. Yet, the *central* surface density has a simple form

$$\sigma_c = (M/2\pi)[(b+M)^2 - a^2](b^2+a^2-M^2)^{-1/2} \times [(b+M)^2+a^2]^{-3/2}.$$

In order for the central density to be positive, one must make the identification at $b^2 > M^2 - a^2$. [This is evident in the black-hole case since the hole's interior is mapped onto the rod $\rho=0$, $-(M^2-a^2)^{1/2} < z < (M^2-a^2)^{1/2}$.] If $a > M$, one has to choose $b > a - M$. The central density can become arbitrarily large for $b^2 \to (M^2 - a^2)^{1/2}$—a region "close" to the horizon (which itself was cut off) is then included. $\sigma_c > 0$ can be large also for $a \gtrsim M$. However, in the cases with $a > M$, it turns out that the physical condition $P \geq 0$ [cf. (12)] leads to the inequality

$$p^2(9x^4-6x^2+1) - p(4x^5-24x^3+4x) - (7x^4-10x^2-1) \leq 0,$$

where $x = b/p$ and $p^2 = a^2/M^2 - 1$ [cf. (3)], which restricts b more strongly. For $a \gg M$ one finds $b_{\min} \sim 9a^2/4M$. Then we can construct physical disks only with $U^*_{\max} \sim 0.3(M/a) \ll 1$.

In coordinates (ρ,z) the ergoregion (cf., e.g., [3]) has a "toroidal" character, the center of the generating circle with φ=const is at $\rho_0 = M(q - 1/2q)$, the radius being $\mathcal{R} = M/2q$. (The ergosphere is the real toroid only for $a > M$; for $a < M$, $\rho_0 < \mathcal{R}$—such set contains the rod representing the hole's interior and is topologically a sphere.) For $1 \geq a/M > 1/\sqrt{2} \doteq 0.707$, the disks given by $(M^2 - a^2)^{1/2} < b < M^2/2a$ produce the ergoregion. Their central density is positive and the graphical results show (see below) that they are physical everywhere. It turns out that physical disks producing an ergoregion exist even for $a/M > 1$, provided that $a/M < 1.044$.

Here we shall confine ourselves to illustrate just one case—with $a/M = 0.8$. The disks corresponding to different choices of b are compared by exhibiting the circular velocities, etc., as functions of the "circumferential" proper radius $\tilde{R} = \sqrt{g_{\varphi\varphi}}$, where $g_{\varphi\varphi}$ is given by (1)-(3). In Fig. 1(a) we give the velocity curves of counter-rotating streams as measured by FIOs. These were calculated, starting from (12), for twelve disks; the disks be-

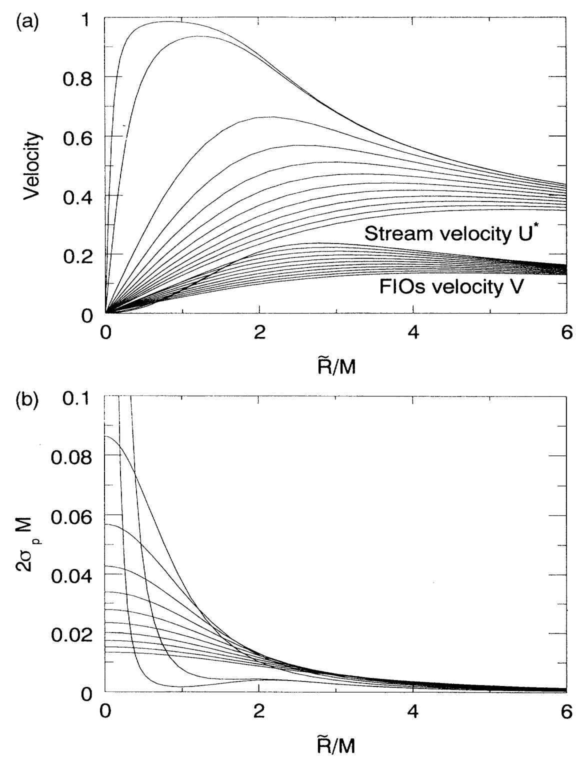

FIG. 1. (a) Velocity curves of particles in the disks as measured by FIOs (upper curves) and of FIOs as measured in LNRFs (lower curves) as functions of proper circumferential radius $\tilde{R}$ for twelve Kerr disks ($a/M = 0.8$, $b/M = 0.6013$, 0.613, 0.87, 1.11, 1.34, 1.56, 1.78, 2.00, 2.21, 2.42, 2.63, 2.84). (b) The sum of proper rest mass densities as a function of $\tilde{R}$ for the same twelve disks as in (a).

come more relativistic with decreasing b. In the highly relativistic disks (upper two curves) U^* increases extremely rapidly as one moves away from the center, approaching the velocity of light at $\tilde{R} \lesssim M$, and then starts decreasing. Also in Fig. 1(a), velocity curves of the FIOs with respect to the LNRFs are plotted by using (15). Although these velocities achieve also high values ($V \sim 0.25$), their maxima occur at larger $\tilde{R} \sim (2\text{-}3)M$. In particular, in the two most relativistic disks, the FIOs [the velocity curves of which cross the other curves from bottom to the top in Fig. 1(a)] do not move so rapidly with respect to LNRFs *close* to the center. (Here "close" refers to the proper circumferential radius $\tilde{R}$.) Just these two disks—from the twelve disks exhibited—produce an ergoregion.

Figure 1(b) gives the plots of the sum of the surface proper rest mass densities of both streams as calculated from (14). For highly relativistic disks, $2\sigma_p$ rises rapidly towards the center; however, it decreases then very rapidly with $\tilde{R}$, and in the most relativistic case even reaches a local minimum at those $\tilde{R}$ where $U^* \to 1$. The central gravitational redshift is $z_c \approx 10^3$ in this case.

We are not aware of any exact analytic solutions for the sources of the stationary gravitational fields with such physical properties as the disks constructed here. Although extending to infinity, they have finite mass and exhibit interesting relativistic properties like high velocities, large redshifts, and dragging effects, including ergoregions. In the Newtonian limit—when no dragging arises—their features are the same as for the rotating disks with only one stream of particles; in the relativistic regime, the central parts of a realistic, highly relativistic, rotating flattened object should have some properties in common with these disks. Incidentally, very recent observations [14] yielded the astonishing result that the disk of the galaxy NGC 4550 is built from two nearly identical *counter*-streaming stellar components; other galaxies are being investigated in light of this discovery.

J.B. is grateful to Donald Lynden-Bell for making possible his visit to Cambridge in 1992 which resulted in the papers [8,10]. After finishing the work presented in this Letter (cf. also [15]) we learned from Professor Lynden-Bell that he and C. Pichon had independently constructed similar Kerr-disks models with radial pressure [16].

[1] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[2] S. Chandrasekhar, in *Highlights of Astronomy* (Reidel, Dortrecht, 1980), Vol. 5, p. 45.
[3] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
[4] A. Krasiński, Ann. Phys. (N.Y.) **112**, 22 (1978).
[5] D. McManus, Classical Quantum Gravity **8**, 863 (1991).
[6] J. M. Bardeen and R. V. Wagoner, Astrophys. J. **167**, 359 (1971).
[7] G. Neugebauer and R. Meinel, Rigidly Rotating Risks of Rust in Reneral Relativity (in German), Abstract in Verh. Dtsch. Phys. Ges., No. 4 (1993); (to be published).
[8] J. Bičák, D. Lynden-Bell, and J. Katz, Phys. Rev. D **47**, 4334 (1993).
[9] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton Univ. Press, Princeton, 1987). The idea first appeared in the work of G. G. Kuzmin, Astron. Zh. **33**, 27 (1956).
[10] J. Bičák, D. Lynden-Bell, and C. Pichon, Mon. Not. R. Astron. Soc. (to be published).
[11] N. W. Evans and P. T. de Zeeuw, Mon. Not. R. Astron. Soc. **257**, 152 (1992).
[12] D. Krammer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, Cambridge, 1980).
[13] C. Barrabès and W. Israel, Phys. Rev. D **43**, 1129 (1991).
[14] V. C. Rubin, J. A. Graham, and J. D. P. Kenney, Astrophys. J. **394**, L9 (1992); H. Rix, M. Franx, D. Fisher, and G. Illingworth, Astrophys. J. **400**, L5 (1992).
[15] J. Bičák and T. Ledvinka, Max-Planck-Institute for Astrophysics, Green Report No. MPA 726 (unpublished).
[16] D. Lynden-Bell and C. Pichon (to be published). (Cf. also the reports by D.L.-B. at the Dedication Symposium, IUCAA, Pune, India, January 1993 and by C.P. at the conference of CNRS, Aussois, April 1993.)