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An Integrable Shallow Water Equation with Peaked Solitons

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We derive a new completely integrable dispersive shallow water equation that is bi-Hamiltonian and thus possesses an infinite number of conservation laws in involution. The equation is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equations in the shallow water regime. The soliton solution for this equation has a limiting form that has a discontinuity in the first derivative at its peak.

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Completely integrable nonlinear partial differential equations arise at various levels of approximation in shallow water theory. Such equations possess soliton solutions—coherent (spatially localized) structures that interact nonlinearly among themselves then reemerge, retaining their identity and showing particlelike scattering behavior. In this paper, we use Hamiltonian methods to derive a new completely integrable dispersive shallow water equation,

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where u is the fluid velocity in the x direction (or equivalently the height of the water's free surface above a flat bottom), κ is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. This equation retains higher order terms (the right-hand side) in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. Dropping these terms leads to the Benjamin-Bona-Mahoney (BBM) equation, or at the same order, the Korteweg-de Vries (KdV) equation. Our extension of the BBM equation possesses soliton solutions whose limiting form as $\kappa \to 0$ has peaks where first derivatives are discontinuous. These "peakons" dominate the solution of the initial value problem for this equation with $\kappa=0$. The way a smooth initial condition breaks up into a train of peakons is by limiting to a verticality at each inflection point with negative slope, from which a derivative discontinuity emerges. Remarkably, the multisoliton solution is obtained by simply superimposing the single peakon solutions and solving for the evolution of their amplitudes and the positions of their peaks as a completely integrable finite dimensional Hamiltonian system.

Our equation is bi-Hamiltonian, i.e., it can be expressed in Hamiltonian form in two different ways. The ratio of its two (compatible) Hamiltonian operators is a recursion operator that produces an infinite sequence of conservation laws. This bi-Hamiltonian property is used to recast our equation as a compatibility condition for a linear isospectral problem, so that the initial value problem may be solved by the inverse scattering transform (IST) method.

The unidirectional model.—Consider Euler's equations for an inviscid incompressible fluid of uniform density with one horizontal velocity component u in the x direction, and w in the vertical (z) direction. The fluid is acted on by the acceleration of gravity, g, and is moving in a horizontally infinite domain with an upper free surface at $z=\zeta(x,t)$ and flat bottom at $z=-h_0$. Substituting the solution form motivated by shallow water asymptotics [1], $u=u(x,t)$, $w=-(z+h_0)u_x$, into the conserved energy (kinetic + potential) for Euler's equations, and explicitly performing the z integration leads to the energy $H_{\rm GN}=\frac{1}{2}\int_{-\infty}^{+\infty}dx[\eta u^2+\frac{1}{3}\eta^3 u_x^2+g(\eta-h_0)^2]$, where $\eta=\zeta+h_0$ is the height of the water above the bottom. Substituting the same solution form above into Euler's equations and integrating over the vertical coordinate leads to the Green-Naghdi (GN) equations [2]. The GN equations conserve the energy $H_{\rm GN}$. In fact, they are expressible in Hamiltonian form [3] as

$$\begin{pmatrix} m_t \\ \eta_t \end{pmatrix} = -\begin{pmatrix} \partial m + m\partial & \eta\partial \\ \partial\eta & 0 \end{pmatrix} \begin{pmatrix} \delta H_{\rm GN}/\delta m \\ \delta H_{\rm GN}/\delta\eta \end{pmatrix}, \tag{2}$$

where the momentum density m is defined by $m = \delta H_{\rm GN}/\delta u$. The GN equations do not necessarily refer to a thin-domain expansion in a small parameter ϵ that measures the ratio of depth to wavelength. In such an expansion the kinetic energy of vertical motion ($\sim \eta^3 u_x^2$) in $H_{\rm GN}$ would be $O(\epsilon^2)$. Shallow water theory makes a further small-amplitude assumption, in the form $\eta = h_0 + O(\alpha)$, $\alpha \ll 1$, and balances $\alpha \simeq \epsilon^2$. In contrast, the Hamiltonian $H_{\rm GN}$ retains nondominant terms (e.g., ζ^3) that would be higher order in such an expansion. Starting from the GN equations, further small-amplitude asymptotics and restriction to unidirectional propagation in a frame moving near the critical wave speed $c_0 = \sqrt{gh_0}$ leads to the KdV equation [4], $u_t + c_0 u_x + \frac{3}{2} u u_x + \frac{1}{6} c_0 \times h_0^2 u_{xxx} = 0$, or, with the same order of accuracy in the thin-domain expansion, the BBM equation [5], $u_t + c_0 u_x + \frac{3}{2} u u_x - \frac{1}{6} h_0^2 u_{xxt} = 0$. Instead of making asymptotic expansions in the equations of motion, as in the derivations of the KdV and BBM equations, our approach in deriving (1) is to make a unidirectional approximation by relating m to η in the GN system and preserving the momentum part of its Hamiltonian structure (2). For this purpose, we will set $\eta = h_0\sqrt{m/h_0 c_0}$, and since $\eta \to h_0$ as $|x| \to \infty$ the boundary conditions on m will be assumed to be $m \to h_0 c_0$ as $|x| \to \infty$. The functional $C = \int_{-\infty}^{+\infty} \sqrt{m}\, dx$ is the Casimir for the Hamiltonian operator $(m\partial + \partial m)$ and so we will refer to this invariant manifold as the Casimir manifold for (2). Next, we scale $u \to \alpha u$ in the Hamiltonian $H_{\rm GN}$, look for m in the form $m = h_0 c_0 + \alpha m_1 + \alpha^2 m_2 + \alpha^3 m_3 + \cdots$ and expand $H_{\rm GN}$ accordingly. With this scaling and expansion, defining m as the variational derivative of the Hamiltonian with respect to u, and balancing at order $O(\alpha^2)$ gives [6] $m_1 = 2(h_0 u - h_0^3 u_{xx}/3)$. The Hamiltonian may then be rewritten as $H_{\rm GN} = H_{\rm 1D} + O(\alpha^3)$, where

$$H_{\rm 1D} = (\alpha^2/4)\int_{-\infty}^{+\infty} m_1 u\, dx + (\alpha/2)\int_{-\infty}^{+\infty} m_1 c_0\, dx,$$

and the factor $\frac{1}{2}$ arises from restricting to a submanifold [7].

The $O(\alpha)$ equation of motion for m on the Casimir manifold is therefore

$$\begin{aligned} m_t &= -(m\partial + \partial m)\delta H_{\rm 1D}/\delta m \\ &= -(\alpha/2)(m\partial + \partial m)u - (c_0/2) m_x , \end{aligned}$$

or, in terms of u,

$$\begin{aligned} u_t - \tfrac{1}{3} h_0^2 u_{xxt} + c_0 u_x + \tfrac{3}{2} \alpha u u_x - \tfrac{1}{6} h_0^2 c_0 u_{xxx} \\ = \tfrac{1}{3} \alpha h_0^2 u_x u_{xx} + \tfrac{1}{6} \alpha h_0^2 u u_{xxx} . \end{aligned} \tag{3}$$

Dropping the right-hand side of this equation gives BBM or KdV, modulo replacing u_{xxt} by $-c_0 u_{xxx}$ [4]. Thus (3) can be seen as a BBM equation extended by retaining higher order terms (selected by the Hamiltonian approach) in an asymptotic expansion in terms of the small-amplitude parameter α. The restriction to the Casimir manifold is equivalent at $O(\alpha)$ to the unidirectionality assumption $\zeta = \sqrt{h_0/g}\, u + O(\alpha)$ in the usual derivation of the KdV and BBM models from the Boussinesq systems [4,8]. In fact, $\zeta = \sqrt{h_0/g}[u - h_0^2/3 u_{xx}] + O(\alpha)$, and in a thin-domain approximation the double derivative term in this expression would acquire a factor ϵ^2.

Rescaling (3), dropping α, and going to a frame of reference moving with speed $\kappa = c_0/4$ reduces the equation to the standard form (1). Notice that (1), like BBM, is not Galilean invariant, i.e., not invariant under $u \to u + \kappa$, $t \to t$, $x \to x + \kappa t$. Thus, Eq. (1) is best seen as a member of a family of equations parametrized by the speed κ of the Galilean frame.

Using the identity $(1 - \partial^2)e^{-|x|} = 2\delta(x)$ and setting $\mathcal{K}[v] \equiv \int_{-\infty}^{+\infty} dy \exp(-|x - y|) v(y)$, expresses Eq. (1) in nonlocal form as

$$u_t + u u_x + 2\kappa \mathcal{K}[u_y] = -\mathcal{K}[u u_y + \tfrac{1}{2} u_y u_{yy}] .$$

Dropping the quadratic terms on the right-hand side of this equation gives the one studied by Fornberg and Whitham [9]. Fornberg and Whitham show that traveling wave solutions of this truncated equation have a peaked limiting form. Moreover, nonsymmetric initial data with two inflection points in their case can develop a vertical slope in finite time.

In a later paper we will discuss the parametrized family (1). The present paper focuses on the limiting case $\kappa = 0$,

$$u_t - u_{xxt} = -3 u u_x + 2 u_x u_{xx} + u u_{xxx} , \tag{4}$$

where u is defined on the real line with vanishing boundary conditions at infinity such that the Hamiltonian $H_1 = \frac{1}{2}\int_{-\infty}^{+\infty}(u^2 + u_x^2)dx$ is bounded. As with (3), H_1 generates the flow (4) through $m = u - u_{xx}$, $m_t = -(m\partial + \partial m)\delta H_1/\delta m$.

Steepening at inflection points.—Consider an initial condition that has an inflection point at $x = \bar{x}$, to the right of its maximum, and decays to zero in each direction sufficiently rapidly for H_1 to be finite. Define the time dependent slope at the inflection point as $s(t) = u_x(\bar{x}(t), t)$. Then the nonlocal form of (4) (with $\kappa = 0$) and standard Sobolev estimates yield a differential inequality for s, $ds/dt \leq -s^2/2 + H_1$. Hence, the slope becomes vertical in finite time, provided it is initially sufficiently negative. If the initial condition is antisymmetric, then the inflection point at $u = 0$ is fixed and $d\bar{x}/dt = 0$, due to the symmetry $(u,x) \to (-u, -x)$ enjoyed by (4). In this case, no matter how small $|s(0)|$, verticality develops in finite time. This steepening property implies that traveling wave solutions of (4) may not have the usual bell shape since inflection points may not be stationary in time. In fact the traveling wave solution is given by $u(x,t) = c\exp(-|x-ct|)$. This solution travels with speed c and has a corner (that is, a finite jump in its derivative) at its peak of height c [10].

N-soliton solution.— Motivated by the form of the traveling wave solution, we make the following solution ansatz for N interacting peaked solutions, $u(x,t) = \sum_{i=1}^{N} p_i(t)\exp[-|x-q_i(t)|]$. Substituting this into Eq. (4) yields evolution equations for q_j and p_j, that are Hamilton's canonical equations, with Hamiltonian H_A given by substituting the solution ansatz above into the integral of motion H_1, yielding $H_A = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j \times \exp(-|q_i - q_j|)$. Hamiltonians of this form describe geodesic motion. The peak position $q_i(t)$ is governed by geodesic motion of a particle on an N-dimensional surface with inverse metric tensor $g^{ij}(\mathbf{q}) = \exp(-|q_i - q_j|)$, $\mathbf{q} \in \mathbb{R}^N$. The metric tensor is singular whenever $q_i = q_j$.

Two-soliton dynamics.— Consider the scattering of two solitons that are initially well separated, and have speeds c_1 and c_2, with $c_1 > c_2$ and $c_1 > 0$, so that they collide. The Hamiltonian system governing this collision possesses two constants of motion, $H_0 = p_1 + p_2 = c_1 + c_2$ and $H_A = (c_1^2 + c_2^2)/2$. Notice that if the peaks were to overlap, thereby producing $q_1 - q_2 = 0$ during a collision, there would be a contradiction $2H_A = (c_1+c_2)^2 = c_1^2 + c_2^2$, unless p were to diverge when the overlap occurred.

The solution of Hamilton's canonical equations for Hamiltonian H_A when $N = 2$ is given by

$$q_1 - q_2 = -\ln\left|\frac{4(c_1-c_2)^2 \gamma e^{(c_1-c_2)t}}{(\gamma e^{(c_1-c_2)t} + 4c_1^2)(\gamma e^{(c_1-c_2)t} + 4c_2^2)}\right|,$$
$$p_1 - p_2 = \pm(c_1 - c_2)\frac{\gamma e^{-(c_1-c_2)t} - 4c_1c_2}{\gamma e^{-(c_1-c_2)t} + 4c_1c_2} \tag{5}$$

and the conservation law for $p_1 + p_2$. Here γ is a constant specifying the initial separation of the peaks, and c_1 and c_2 are the asymptotic $t \to \pm\infty$ values of their speeds, or amplitudes. The divergence of p_1 and p_2 in Eq. (5) associated with soliton overlap can only occur when c_1 and c_2 have opposite signs. That is, only "head-on" collisions can lead to overlapping peaks (see Fig. 1 for the "soliton-antisoliton" case $c_1 = -c_2 = c$).

The two soliton solution (5) determines the "phase shifts," i.e., the shifts in the asymptotic position for $t \to \infty$, that the solitons experience after interaction. As $t \to +\infty$ the solitons reemerge unscathed, the faster (and larger) soliton ahead of the slower (and smaller) one. Defining the phase shift for the faster soliton to be $\Delta q_f \equiv q_2(+\infty) - q_1(-\infty)$, and for the slower soliton, $\Delta q_s \equiv q_i(+\infty) - q_2(-\infty)$, leads to $\Delta q_f = \ln[c_1^2/(c_1-c_2)^2]$, and $\Delta q_s = \ln[(c_1-c_2)^2/c_2^2]$. These formulas show that when $c_1/c_2 > 2$ both solitons experience a forward shift.

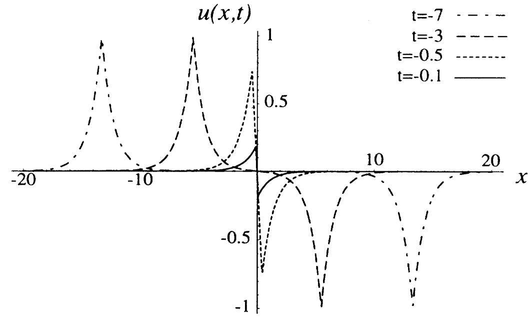

FIG. 1. The soliton-antisoliton solution u reconstructed from Eq. (5) is $u(x,t) = c(\exp[-|x - \frac{1}{2}q(t)|] - \exp[-|x + \frac{1}{2}q(t)|])/\tanh(ct)$. This solution displays the steepening behavior discussed in the text. The slope becomes vertical and the amplitude of the solution becomes (everywhere) zero right at the moment of overlap. At later times the peaks redevelop and depart again according to the symmetry $(u,t) \to (-u,-t)$.

For $1 < c_1/c_2 < 2$ the faster soliton is shifted forward while the slower soliton is shifted backward. When $c_1/c_2 = 2$ no shift occurs for the slower soliton.

Bi-Hamiltonian structure.— Equation (4) follows, as well, from an action principle expressed in terms of a velocity potential. This action principle leads to an additional conserved quantity, $H_2 = \frac{1}{2}\int_{-\infty}^{+\infty}(u^3 + uu_x^2)dx$, and *another* Hamiltonian operator, $\partial - \partial^3$. Our Eq. (4) then can be written in Hamiltonian form in two different ways, $m_t = -(\partial - \partial^3)\delta H_2/\delta m = -(m\partial + \partial m)\delta H_1/\delta m$. The two Hamiltonian operators $B_1 = \partial - \partial^3$ and $B_2 = \partial m + m\partial$ form a Hamiltonian pair. That is, their sum is still a Hamiltonian operator [11]. Equation (4) is thus bi-Hamiltonian and has an infinite number of conservation laws recursively related to each other by $B_1\delta H_n/\delta m = B_2\delta H_{n-1}/\delta m \equiv -m_t^{(n+1)}$, $n = 0, \pm 1, \pm 2, \ldots$. Starting from H_1 and H_2 this relation generates an infinite sequence of conservation laws including, e.g., $H_0 = \int_{-\infty}^{+\infty} m\, dx$, $H_{-1} = \int_{-\infty}^{+\infty}\sqrt{m}\, dx = C$, $H_{-2} = \frac{1}{2}\int_{-\infty}^{+\infty}[m_x^2/4m^{5/2} - 2/\sqrt{m}]dx$, etc. Correspondingly, the recursion operator $\mathcal{R} = B_2 B_1^{-1}$ generates a hierarchy of commuting flows, defined by $m_t^{(n+1)} = K_{n+1}[m] = \mathcal{R}K_n[m]$, $n = 0, \pm 1, \pm 2, \ldots$. The first few flows in the hierarchy are $m_t^{(0)} = -(\partial - \partial^3)(2\sqrt{m})^{-1}$, $m_t^{(1)} = 0$, $m_t^{(2)} = -m_x$, and $m_t^{(3)} = -(m\partial + \partial m)u$. The last of these is our Eq. (4) and the first is an extension of the integrable Dym equation [12]. It turns out that all the flows in this hierarchy are isospectral and thus completely integrable.

The isospectral problem.— In order to find the isospectral problem for our equation, we follow Gel'fand and Dorfmann [13] in considering the skew symmetric spectral problem, $(\lambda B_1 - B_2)\phi = 0$. A class of solutions of this problem are related by $\phi = \psi^2$ to the solutions ψ of a second order symmetric spectral problem. By imposing isospectrality, $\lambda_t = 0$, our Eq. (1) follows from the compatibility condition $\psi_{xxt} = \psi_{txx}$ of the system for $\psi(x,t)$,

$$\psi_{xx} = \left[\frac{1}{4} - \frac{m(x,t)+\kappa}{2\lambda}\right]\psi\,, \qquad \qquad (6)$$
$$\psi_t = -(\lambda+u)\psi_x + \tfrac{1}{2}u_x\psi\,.$$

This is the isospectral problem we seek. The system (6) provides a means of solving the initial value problem for (1) by the purely linear inverse-scattering transform technique [12]. For instance, if the boundary conditions on m are taken to be zero at $x=\pm\infty$ (sufficiently fast) [14], then the spectral problem (6) when $\kappa=0$ has a purely discrete spectrum since $\psi(x)\to\exp(\pm x/2)$ as $|x|\to\infty$; i.e., eigenfunctions always decay exponentially at infinity. If, e.g., the initial condition $u(x,0)$ is chosen such that $u(x,0)=A[(\pi/2)e^x-2\sinh x\arctan(e^x)-1]$, so that $m(x,0)=A\,\mathrm{sech}^2(x)$, for an arbitrary constant A, then it is easy to show [15] that the eigenvalues λ for (6) are given by $\lambda_n=2A/[(2n+1)(2n+3)]$, $n=0,1,2,\ldots$. This formula shows explicitly that $\lambda=0$ is an accumulation point for the discrete spectrum and the eigenvalues converge to it as $1/n^2$, $n\to\infty$, a fact that holds in general for any initial condition decaying exponentially fast at infinity. Equations (6) also imply that the N-soliton mechanical system with Hamiltonian H_A is completely integrable [16]. When $\kappa\neq0$, i.e., for an equation in the family (1), the limiting behavior of ψ becomes $\psi(x)\to\exp(\pm x\sqrt{1/4-\kappa/2\lambda})$ as $|x|\to\infty$, and so a continuous spectrum develops out of the origin in the interval $0\le\lambda\le2\kappa$. Also, for $\kappa\neq0$ the soliton solution of (1) becomes C^∞ and there is no derivative discontinuity at its peak. The peculiar feature of the disappearance of the continuous spectrum in the limit $\kappa\to0$ can be traced to the constant $\frac{1}{4}$ in the spectral problem (6), which in turn is generated by the first derivative operator in B_1.

Numerical simulations [17] confirm the analysis discussed here and demonstrate the robustness of the peaked soliton solutions. These simulations clearly illustrate the inflection point mechanism by which a localized (positive) initial condition breaks up into a height-ordered train of peaked solitons moving to the right, with the tallest ones ahead.

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