Evidence for Two Exponent Scaling in the Random Field Ising Model

Michael Gofman,[1] Joan Adler,[2,3] Amnon Aharony,[3] A. B. Harris,[4] and Moshe Schwartz[3]
[1]School of Mathematical Sciences, Beverly and Raymond Sackler Faculty of Exact Sciences, Tel Aviv University, 69978, Tel Aviv, Israel
[2]Department of Physics, Technion-IIT, 32000, Haifa, Israel
[3]School of Physics and Astronomy, Beverly and Raymond Sackler Faculty of Exact Sciences, Tel Aviv University, 69978, Tel Aviv, Israel
[4]Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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Novel methods were used to generate and analyze new 15 term high temperature series for both the (connected) susceptibility χ and the structure factor (disconnected susceptibility) χ_d for the random field Ising model with dimensionless coupling $K=J/kT$, in general dimension d. For both the bimodal and the Gaussian field distributions, with mean square field J^2g, we find that $(\chi_d-\chi)/K^2g\chi^2=1$ as $T\rightarrow T_c(g)$, for a range of $[h^2]=J^2g$ and $d=3,4,5$. This confirms the exponent relation $\bar{\gamma}=2\gamma$ (where $\chi_d\sim t^{-\bar{\gamma}}$, $\chi\sim t^{-\gamma}$, $t=T-T_c$) proving that random field exponents are determined by two (and not three) independent exponents. We also present new accurate values for γ.

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The critical behavior of random field systems has been the subject of intense research since the pioneering work of Imry and Ma [1]. For a while it seemed that the problem had been solved [2–4] using the idea of dimensional reduction: Diagrammatic expansions imply that the exponents of the random field system in d dimensions are equal to those of the pure system in $d-2$ dimensions. However, accumulated evidence [5,6] has forced the community to abandon dimensional reduction in favor of more accurate theories. Basically, all of these theories generalize the idea of dimensional reduction, in that hyperscaling relations (which involve the dimensionality) for the random field system contain the shifted value $d-\theta$ instead of d. However, the literature contains an open controversy concerning the exponent θ, which describes the singular free energy F_ξ in a correlation volume [of the size of the correlation length $\xi\sim(T-T_c)^{-\nu}$], $F_\xi\sim\xi^\theta$. One approach [7–9], which follows a conjecture put forward already in Ref. [2], maintains that $F_\xi\sim g\chi$, where $g\sim[h^2]$ is the mean square field and χ is the susceptibility. Since $\chi\sim\xi^{2-\eta}$, one finds that $\theta=2-\eta$. Having related θ to η, this approach needs only two independent critical exponents (e.g., η and ν) in order to derive all the other exponents. In contrast, an alternative approach [10–13] started from a droplet picture and maintained that θ is a new independent exponent, so that one needs three independent exponents.

The Hamiltonian of the random field Ising model is written as

$$\mathcal{H}=-J\left[\sum_{\langle ij\rangle}\sigma_i\sigma_j+\sum_i h_i\sigma_i\right], \tag{1}$$

where $\langle ij\rangle$ denotes nearest neighbor pairs and $\sigma_i=\pm1$. The quenched random fields h_i are uncorrelated, with $[h_i]=0$ and $[h_ih_j]=J^2g\delta_{ij}$, where $[\,]$ denotes the configurational quenched average. Commonly used distributions are the bimodal ($h_i=\pm J\sqrt{g}$) and the Gaussian. One then expects a critical line in the T-g plane, with the critical temperature $T_c(g)$ decreasing as g increases. Since g is relevant, any finite length series should exhibit a crossover from the nonrandom behavior to that of the random field one. At large g, mean-field theory [14] showed that the bimodal distribution has a first order transition, beyond a tricritical point. This leaves an intermediate window of g values in which critical behavior of the random field type is expected. Existing numerical values for the exponents are uncertain: Estimates from simulations [15] tend to support the two exponent picture. Some other numerical approaches do not shed light on this issue but calculate some of the critical exponents. These find the exact ground state for a finite random field Ising system [16,17] or use real space renormalization [18] for given field configurations. In both cases it is necessary to average over a finite number of field configurations. The droplet picture [10–13] also yields numerical values for exponents, but these involve somewhat uncontrolled approximations (e.g., different expressions for ν in Refs. [10] and [11]).

A direct way to address the issue of two versus three independent exponents involves the difference between the susceptibility,

$$\chi=\sum_j[\langle\sigma_0\sigma_j\rangle-\langle\sigma_0\rangle\langle\sigma_j\rangle] \tag{2}$$

(here, $\langle\,\rangle$ denotes the thermal average, for a fixed quenched configuration) and the structure factor (measured, e.g., in scattering experiments),

$$\chi_d=\sum_j[\langle\sigma_0\sigma_j\rangle]\,. \tag{3}$$

χ and χ_d involve sums over connected and disconnected correlation functions, respectively. For nonrandom systems, $\chi_d=\chi\sim(T-T_c)^{-\gamma}$ for $T>T_c$. In random systems for $T>T_c$, $\chi_d\sim(T-T_c)^{-\bar{\gamma}}$, with $\bar{\gamma}=\nu(4-\bar{\eta})\neq\gamma$.

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In fact, the droplet model [10–13] yields the scaling relation

$$\theta = 2 + \eta - \bar{\eta} , \tag{4}$$

so that one can replace θ by $\bar{\eta}$ (or $\bar{\gamma}$) as the third independent exponent. However, Ref. [7] proposed the additional relation $\bar{\eta} = 2\eta$, which also implies $\bar{\gamma} = 2\gamma$ and $\theta = 2 - \eta$. In fact, a detailed proof [8,9] required an even stronger relation to hold, namely, that

$$A = \lim_{T \to T_c^+} \frac{\chi_d - \chi}{K^2 g \chi^2} = 1 , \tag{5}$$

where $K = J/kT$. In addition to giving the relation $\bar{\gamma} = 2\gamma$, Eq. (5) also requires a unique universal value for the amplitude ratio A.

In this Letter we present very precise numerical evidence that Eq. (5) holds, for both the bimodal and Gaussian random field distributions and for dimensions $d = 3$, 4, 5, and 8. In addition to this general statement, we also present new accurate estimates for the critical exponents γ and $\bar{\gamma}$ in these dimensions. In $d = 3$, 4, and 5, we observe finite-temperature nontrivial critical behavior, and in $d = 8$ mean-field behavior is seen. Details, as well as results for other mean-field dimensions $d \geq 6$, and for $d = 2$, where no finite temperature transition is seen, will be published separately. Our new results are based on new fifteenth order exact high temperature series expansions for both distributions, in general d. As we explain below, the achievement of these results was made possible due to several technical advances in both the generation [extending earlier work using no-free-ends (NFE) diagrams [19,20]] and the analysis (using new visualization techniques [21]) of the series. The former allowed us to (a) increase the number of terms for χ from [22,23] 7 to 15 and (b) generate completely new series for χ_d. The latter allowed us to analyze efficiently hundreds of series for different values of d and g and obtain accurate estimates for T_c, A, γ, and $\bar{\gamma}$.

Series expansions have a great advantage over Monte Carlo simulations, since they do not suffer from metastability and equilibration problems. However, significant advances in the length of the series involve both algorithmic complexity and long computer times. Once these are overcome, our approach [19] generates double series in $K = J/kT$ (or $w = \tanh K$) and g with coefficients which are known polynomials in d. We used MATHEMATICA for the complicated manipulations in generating the series, for collecting the diagrammatic data into the series coefficients, and for checks of part of the FORTRAN code. The actual generation of the series was based on calculating the partition function as a series in w and in $\tau_i = \tanh(h_i/kT)$, taking derivatives with respect to h_i and finally expanding in powers of h_i and averaging.

We started by generating 15 terms for the nonrandom Ising model in general d (requiring 842 NFE diagrams) [24]. Our next important advance is based on the observation that since $w = \tanh K = K + O(K^3)$, diagrams with n bonds generate g-independent terms of leading order K^n but g-dependent terms of leading order $K^{n+2}g$. Therefore, the last two terms in the series (of orders K^{n-1} and K^n) contain only nonrandom diagrams, plus g-dependent contributions from diagrams with up to $n-2$ bonds. Having already generated 15 terms for the nonrandom parts, we only needed to use full (non-NFE) diagram lists (20724 diagrams) to thirteenth order to obtain the full random field series to fifteenth order. We compared MATHEMATICA and FORTRAN results for some thirteenth order diagrams with several loops.

The generation of the series took about six months on a dedicated IBM 320 RISC workstation, and details will be published separately. We studied hundreds of series for different values of d and g and the two field distributions [25]. Our analysis was divided into four stages. We first used a biased dlog Padé analysis of series obtained from term-by-term-division [26] of the coefficients of the series for χ_d by those of χ. Such series diverge as $(1-K)^{\bar{\gamma}-\gamma+1}$. The resulting approximate estimates for $\bar{\gamma}-\gamma$ showed a very rapid increase (at $g < 0.1$) from zero (at $g = 0$) to values of $\bar{\gamma}-\gamma$ which are close to estimates of γ found by later direct analyses (see below). As g is increased further, $\bar{\gamma}-\gamma$ exhibits a very slow increase over a wide range in g. This range of almost a plateau is much larger than observed before with the much shorter series [22,23]. At still larger g we saw a second crossover, with an apparent faster increase in g. This may signal the approach to the tricritical point. We have thus concentrated on the "plateau" region, $0.1 < g < g_1$, with $g_1 = 1.25$, 6, and 15 (Gaussian) and $g_1 = 1.25$, 4, and 10 (bimodal) at $d = 3$, 4, and 5, respectively.

In the second stage, we combined our recently developed efficient three dimensional visualization methods [21] with several analysis algorithms [19], which allow nonanalytic confluent corrections to scaling, to study series for χ and χ_d in the above g windows. Values of the critical values $K_c(g)$ at selected g values are given in Table I, and we discuss the exponent values below.

In the third stage, we obtained Padé approximants for A at $K_c(g)$ (as obtained above). As found in other studies [19,27] such a determination of ratios which involve only amplitudes on the same side of the transition is very stable to errors in K_c and in correction terms. Table I also exhibits several high near-diagonal approximants for A. Clearly, $A = 1$ to better than ± 0.003 throughout the plateau range in $d = 3$. Equation (5) has thus been confirmed. Note that the "plateau" in A is flatter than that in the exponents, and therefore the amplitude ratio is a better test of universality (which requires that A, γ, and $\bar{\gamma}$ should be independent of g). A also started to increase for $g > g_1$.

Finally, we deduced overall exponent estimates. Having identified the range of g values for which A is practically constant, we looked back at the values of γ and $\bar{\gamma}$

TABLE I. Values of the amplitude ratio A [Eq. (5)] for selected choices of d and g.

g	Distribution	K_c	Approximant [6/6]	[5/6]	[6/5]	[5/5]
	$d=3$					
0.15	Gaussian	0.2268	0.999866	0.999899	0.999899	0.999866
0.75	Gaussian	0.24675	0.997313	0.997795	0.997791	0.997313
0.15	Bimodal	0.2267	0.999865	0.999898	0.999898	0.999865
0.75	Bimodal	0.2478	0.997377	0.997845	0.997842	0.997377
	$d=4$					
3.5	Gaussian	0.170625	1.01077	1.04432	1.04229	1.01077
6	Gaussian	0.1895	1.03759	1.03708	1.03708	1.03759
3	Bimodal	0.1691	1.00823	1.03072	1.02979	1.00823
	$d=5$					
8	Gaussian	0.1315	1.01201	1.01200	1.01200	1.01201
10	Gaussian	0.136875	1.01913	1.01836	1.01835	1.01913
5	Bimodal	0.125313	1.00486	1.00574	1.00572	1.00486
8	Bimodal	0.134918	1.01344	1.01343	1.01343	1.01344

measured at the second stage. The series for χ_d contain [8,9] more correction terms and generally behave less well than those for χ. [As seen from Eq. (5), χ_d contains a contribution of order χ, which acts as a correction to the leading behavior. In addition, the numerator approaches the denominator only asymptotically; for finite $T-T_c$ the difference between the two also adds singular corrections to χ_d. Both of these do not arise in the series for χ.] Given our results that $A=1$, we already know that $\bar{\gamma}=2\gamma$, and therefore we quote only values for γ. Averaging over the gradual increase in the exponents with g, and including the appropriate range in the error bars, we quote for both distributions, $\gamma=2.1\pm0.2$, 1.45 ± 0.05, and 1.13 ± 0.02 for $d=3$, 4, and 5. For $d=8$ we found $2\gamma=\bar{\gamma}=2.00\pm0.01$. Our $d=3$ value of γ can be compared with those found from (a) determination of the exact ground state for a finite system, $\bar{\gamma}=2.9$ [16], $\gamma=1.58-1.6$ [17], (b) real space renormalization [18], $\gamma=1.9-2.2$, (c) modified dimensional reduction [28], $\gamma=2.05-2.2$, $\bar{\gamma}=2\gamma$, and (d) older series expansions, $\gamma=1.42-1.48$ [23] and $\gamma=1.7$ [22]. In $d=5$, we can compare only with the older series results of [22], $\gamma=1.08-1.18$ and [23] $\gamma=1.17-1.19$. Pleasing agreement is seen with Refs. [18] and [28], and the $d=5$ result of Ref. [22].

In conclusion, we have shown that there are only two independent random field exponents. While our study is decisive on this matter, further simulation studies remain of interest, and in particular a study of the ratio A. Our precise critical temperature estimates should be a useful input to such studies. We also note that our results may now be compared with real experiments, since magnetic x-ray scattering seems to show real equilibrium very close to the surface of dilute antiferromagnets in a field [29].

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[1] Y. Imry and S. K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[2] A. Aharony, Y. Imry, and S. K. Ma, Phys. Rev. Lett. **37**, 1364 (1976).
[3] A. P. Young, J. Phys. C **10**, L257 (1977).
[4] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[5] G. Grinstein and S. K. Ma, Phys. Rev. Lett. **49**, 685 (1982).
[6] J. Villain, J. Phys. (Paris) **43**, L551 (1982).
[7] M. Schwartz, J. Phys. C **18**, 135 (1985).
[8] M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985); Phys. Rev. B **33**, 2059 (1986).
[9] M. Schwartz, M. Gofman, and T. Natterman, Physica (Amsterdam) **178A**, 6 (1991).
[10] J. Villain, J. Phys. (Paris) **46**, 1843 (1985).
[11] A. J. Bray and M. A. Moore, J. Phys. C **18**, L927 (1985).
[12] D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986); note that the η in this reference is equal to our $\eta-2$.
[13] T. Nattermann, Phys. Status Solidi (b) **131**, 563 (1985).
[14] A. Aharony, Phys. Rev. B **18**, 3318 (1978).
[15] A. T. Ogielski and D. A. Huse, Phys. Rev. Lett. **56**, 1298 (1986); H. Rieger and A. P. Young (to be published).
[16] A. T. Ogielski, Phys. Rev. Lett. **57**, 1271 (1986).
[17] H. E. Cheung, Phys. Rev. B **33**, 6191 (1986).
[18] I. Dayan, M. Schwartz, and A. P. Young (to be published).
[19] Many aspects of this calculation (including the improved convergence as d increases) are common to other general-d series projects discussed in J. Adler, Y. Meir, A. Aharony, A. B. Harris, and L. Klein, J. Stat. Phys. **58**, 511 (1990), and references therein.
[20] A. B. Harris, Phys. Rev. B **36**, 337 (1987). The diagram listings can be found in A. B. Harris and Y. Meir, Phys. Rev. B **36**, 1840 (1987).
[21] J. Adler, I. Chang, and S. Shapira (to be published).
[22] Y. Shapir and A. Aharony, J. Phys. C **15**, 1361 (1982).
[23] A. Houghton, A. Khurana, and F. J. Seco, Phys. Rev. B **34**, 1700 (1986).

[24] The analysis of these series [C. Munkel, D. Heermann, J. Adler, M. Gofman, and D. Stauffer, Physica (Amsterdam) **193A**, 540 (1993); M. Gofman, J. Adler, A. Aharony, A. B. Harris, and D. Stauffer (to be published)], yielded results which were in full agreement with (and in some cases even more precise than) simulations, thus serving as a severe check on our programs.

[25] The series themselves are available upon request. To enable alternative analyses, we quote the series for a Gaussian distribution with $d=3$, $g=0.75$:

$$\chi = 1 + 6K + 29.25K^2 + 139K^3 + 635.1250000000001K^4 + 2885.675K^5 + 12857.06770833333K^6 + 57081.66369047622K^7 + 250663.8218005952K^8 + 1.098038103786377\times 10^6 K^9 + 4.774814962634355\times 10^6 K^{10} + 2.072326948508954\times 10^7 K^{11} + 8.94547059804297\times 10^7 K^{12} + 3.85528959145134\times 10^8 K^{13} + 1.654504520887405\times 10^9 K^{14} + 7.09072307593506\times 10^9 K^{15}$$

and

$$\chi_d = 1 + 6K + 30K^2 + 148K^3 + 706K^4 + 3357.425K^5 + 15702.70833333333K^6 + 73228.30119047623K^7 + 338307.7327380951K^8 + 1.558629230572091\times 10^6 K^9 + 7.134352148627654\times 10^6 K^{10} + 3.257002101889419\times 10^7 K^{11} + 1.479494567738371\times 10^8 K^{12} + 6.703881287502365\times 10^8 K^{13} + 3.184453242841873\times 10^9 K^{14} + 1.45198131568335\times 10^{10} K^{15}.$$

[26] D. L. Hunter and G. A. Baker, Phys. Rev. B **7**, 3346 (1973); Y. Meir, J. Phys. A **20**, L349 (1987).

[27] J. Adler, A. Aharony, Y. Meir, and A. B. Harris, J. Phys. A **19**, 3631 (1986).

[28] M. Schwartz, J. Phys. C **21**, 8531 (1988).

[29] J. P. Hill, T. R. Thurston, R. W. Erwin, M. J. Ramstad, and R. J. Birgeneau, Phys. Rev. Lett. **66**, 3281 (1991).