Cascade Model for Intermittency in Fully Developed Magnetohydrodynamic Turbulence

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Evidence of intermittency in the interplanetary space plasma has been recently pointed out. We present a magnetohydrodynamic (MHD) cascade model, derived from a binomial process, whose results fit the observed scaling law for the qth power of the structure functions. Using this model we modify the oldest Kraichnan theory showing, for the first time, the multifractal structure of fully developed MHD turbulence.

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In an early work on fully developed magnetohydrodynamic (MHD) turbulence, by using the hypotheses that (i) the nonlinear interactions occur only between Alfvénic fluctuations of the same length scale l and (ii) the energy flux from vortices of scales $>l$ to vortices of scales $<l$ assumes a value independent from l (no intermittency), Kraichnan [1] derived the $k^{-3/2}$ ($k \sim l^{-1}$) power law for the inertial range spectrum of the magnetic and kinetic energy densities in the stationary state. The difference with the $k^{-5/3}$ Kolmogorov spectrum for hydrodynamic turbulence is due to the fact that [1,2] the nonlinear interactions in MHD are slowed down, since they occur only between fluctuations propagating in opposite directions with respect to the magnetic field of the largest scale. Recently, analyzing satellite observations in the solar wind, some authors [3-5] showed that the exponents of the qth power of the structure functions of the measured velocity and magnetic fields are nonlinear function of q. These observations indicate that [6] intermittency is present in MHD fluctuations and, as a consequence, the Kraichnan theory, which describes fully developed MHD turbulence, should be modified. Up to now, whereas many works have been published about intermittency modification of the Kolmogorov theory [6], nothing has been done for the Kraichnan theory.

The equations describing incompressible MHD turbulence can be written as

$$\frac{\partial \mathbf{Z}^{\pm}}{\partial t} + (\mathbf{Z}^{\mp} \cdot \nabla)\mathbf{Z}^{\pm} = -\frac{1}{\rho}\nabla P, \tag{1}$$

P being the total (magnetic plus kinetic) pressure and ρ the plasma mass density. The variables $\mathbf{Z}^{\pm} = \mathbf{v} \pm \mathbf{B}/\sqrt{4\pi\rho}$ ($\mathbf{v}$ and $\mathbf{B}$ are, respectively, the velocity and the magnetic fields) represent the two possible Alfvénic modes propagating in opposite directions along the magnetic field of the largest scale. Equations (1) contain neither source nor dissipative terms as we concentrate upon the inertial range of the turbulence, i.e., on scale lengths $l_D \ll l \ll L$ (l_D is the length at which dissipative effects become important, and L is the greatest length scale). From a physical point of view the existence of the inertial range, i.e., the excitation of smaller and smaller length scales l, can be viewed as a nonlinear energy cascade from the scale L to l_D. This energy cascade [2] is realized stochastically in a time $T^{\pm} \sim (\tau_{\rm NL}^{\pm})^2/\tau_A$, due to $N^{\pm} \sim (\tau_{\rm NL}^{\pm}/\tau_A)^2$ statistical encounters with random phases between eddies of the same scale length l and amplitudes $Z^{\pm}(l)$ [being $Z^{\pm}(l)$ an order of magnitude estimate for $\mathbf{Z}^{\pm}$]. The time $\tau_{\rm NL}^{\pm} \sim l/Z^{\mp}$ represents the lifetime of the nonlinear interactions, while $\tau_A \sim l/c_A$ is the time needed to decorrelate the interacting eddies (c_A being the Alfvén velocity related to the scale L). The energy transferred per unit time towards the smaller scales $\Pi^{\pm}(l) \sim (Z^{\pm})^2/T^{\pm}$ is then given by

$$\Pi^{\pm}(l) \sim \frac{(Z^{\pm})^2 (Z^{\mp})^2}{c_A l} \tag{2}$$

and turns out to be the same for both the $Z^{\pm}$ modes.

The MHD equations (1) are invariant [7] under the following scaling transformations:

$$l \to l'\lambda^{-1}, \quad Z^{\pm} \to (Z^{\pm})'\lambda^{-h},$$

where $\lambda > 0$ and h is a free parameter (the times scale as $t \to t'\lambda^{h-1}$, and we choose $P \to P'\lambda^{2-2h}$). Two main consequences of this invariance are the following: (i) For each value of h the quantity $Z^{\pm}/l^h$ is also invariant, so that we can expect to obtain a scaling law where $Z^{\pm}$ scales as l^h. (ii) The energy transfer rate scales as $\Pi^{\pm} \to (\Pi^{\pm})'\lambda^{1-4h}$. Then $\Pi^{\pm}$ assumes a value independent on the scale l (no intermittency) only if $h = \frac{1}{4}$, and in this case the qth power of the structure functions, defined by $\langle |\Delta z^{\pm}|^q \rangle = \langle |z^{\pm}(x+l) - z^{\pm}(x)|^q \rangle$ (brackets being spatial averages and $z^{\pm} = Z^{\pm}/c_A$), scales as $\langle |\Delta z^{\pm}|^q \rangle \sim (l/L)^{q/4}$. By defining the pseudoenergy density spectra $E^{\pm}(k)$ through $\frac{1}{2}\langle (\mathbf{Z}^{\pm})^2 \rangle = \int dk\, E^{\pm}(k)$, the $q/4$ scaling law leads to the Kraichnan spectra $E^{\pm}(k) \sim k^{-3/2}$. On the contrary, in the presence of intermittency, $\Pi^{\pm}$ can be considered as a spatially fluctuating quantity, or, in other words, the singularity h can assume any value [7]. We then introduce, for the structure functions, a scaling law of the type $\langle |\Delta z^{\pm}|^q \rangle \sim (l/L)^{\xi(q)}$, and assuming $Z^+ \sim Z^-$, from (2) we find

$$\langle |\Delta z^{\pm}|^q \rangle \sim (l/L)^{q/4} \langle |\epsilon^{\pm}|^{q/4} \rangle, \tag{3}$$

where $\epsilon^{\pm} = \Pi^{\pm}(l) L/c_A^3$. To obtain an expression for

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$\xi(q)$ we introduce, for the energy transfer rate, a set of singularities α defined by $\epsilon^{\pm}(l)\sim(l/L)^{\alpha}$ in the limit $l/L \to 0$. These singularities are related to h through $\alpha=h-\frac{1}{4}$, so that assuming for the qth power of the energy transfer the following scaling law, $\langle|\epsilon^{\pm}|^{q}\rangle\sim(l/L)^{\varphi(q)}$, from Eq. (3) we found the relation $\xi(q)=q/4+\varphi(q/4)$.

In order to find the curve $\varphi(q)$, we introduce a simple cascade model corresponding to a binomial process, represented as a generalized two-scale Cantor set with equal partition intervals [6,8]. Let us suppose that an eddy of size l, with an assigned measure $\Pi^{\pm}(l)$, breaks up into two eddies of equal size $l/2$, and let us suppose that the energy flux proceeds unequally on the two eddies of size $l/2$. A simple choice for the transfer is to assume that one of the smaller eddies (selected randomly) receives a fraction $p\Pi^{\pm}(l)$ and the other eddy receives a fraction $(1-p)\Pi^{\pm}(l)$. This choice preserves the measure $\Pi^{\pm}$, since we are dealing with inertial range quantities, so that dissipative effects are negligible. This fragmentation process, starting at the length L, is repeated for smaller and smaller length scales, until the end of the inertial range where the energy is dissipated and the eddies' fragmentation is stopped. The parameter $0<p<1$ is a measure of the asymmetry of the energy flux towards the smaller eddies: for $p=\frac{1}{2}$ there is no intermittency since the energy cascade is symmetric, whereas for extreme values of p the energy flux is strongly asymmetric. After n steps of the cascade, the size of each eddy is $l/L=2^{-n}$, and there are $\binom{n}{m}$ eddies where $\epsilon^{\pm}=p^{n-m}(1-p)^{m}$ $(m=0,1,\ldots,n)$. In the limit $n\to\infty$, the number of eddies is given approximately by $\exp\{-[m-n/2]^2/(4n)^{1/2}\}$ which is strongly peaked around $m=n/2$ and has a width of the order of $\sqrt{n}$. These eddies receive almost the same fraction of the measure, say $\delta=p^{n/2}(1-p)^{n/2}$, and they occupy a total length $\beta^n=\delta 2^n=\{2[p(1-p)]^{1/2}\}^n$. If $p\neq\frac{1}{2}$ then $\beta<1$, so that the energy flux is concentrated asymptotically on a limit set whose volume tends to zero as $n\to\infty$, while its fractal dimension can be estimated as the Hausdorff dimension $D_H=2+\ln 2/\ln[p(1-p)]^{-1/2}$. Summation over all eddies at a given scale l yields $\sum|\epsilon^{\pm}(l)|^{q}=[p^{q}+(1-p)^{q}]^{n}$, and using the identity [9] $\sum|\epsilon^{\pm}(l)|^{q}=(l/L)^{(q-1)D_q}$, we found $\varphi(q)=(q-1)D_q$. The dimensions

$$D_q=\log_2[p^q+(1-p)^q]^{1/(1-q)}$$

correspond to the generalized dimensions set of Hentschel and Procaccia [9]. The multifractal structure of MHD turbulence can be described by introducing the so-called singularity spectrum [6,8] $f(\alpha)$, given in our model by

$$f(\alpha)=\log_2(n/m)-(1-m/n)\log_2(n/m-1)\,.$$

This function [8] represents the set of fractal dimensions related to each singularity

$$\alpha=-(m/n)[\log_2 p+(n/m-1)\log_2(1-p)]\,.$$

These results refer to a one-dimensional cut, while the corresponding three-dimensional formulas can be obtained simply [6] by adding 2 to α, $f(\alpha)$, and D_q. Using these results we finally obtain, for the scaling exponent of the structure functions of $Z^{\pm}$, the following expression:

$$\xi(q)=1-\log_2[p^{q/4}+(1-p)^{q/4}]\,, \tag{4}$$

which is valid in the limit of $Z^{+}\sim Z^{-}$ (standard MHD turbulence, in the terminology of Ref. [10]). The Kraichnan scaling law is obviously recovered when $p=\frac{1}{2}$, whereas, if intermittency is present, the pseudoenergy density spectra are $E^{\pm}(k)\sim k^{-m^{\pm}}$, with spectral indices given by $m^{\pm}=\frac{3}{2}+B(p)$. The function $B(p)=\xi(2)-\frac{1}{2}$ represents the modification to the classical Kraichnan spectral index. Note that, since $B(p)\geq 0$ for each p, intermittency gives rise to spectra steeper than the Kraichnan spectra. Furthermore it is worthwhile to note that, if the energy flux is strongly asymmetric, we find $m^{\pm}\sim\frac{5}{3}$; i.e., the pseudoenergy densities can follow a Kolmogorov-like spectrum also in the MHD framework. Up to now this particular consequence of intermittency in MHD, which perhaps could have some interesting consequences in understanding the origin of low-frequency fluctuations in the interplanetary medium [10], has not been investigated.

We have fitted the curve (4), through a χ^2 method, on the corresponding values found by Burlaga [3] in his analysis of Voyager measurements at 8.5 AU (astronomical units). We found for the intermittency parameter the best fit $p=0.69\pm0.01$. Using this value the end points of the curve $f(\alpha)$ are $D_{-\infty}\sim0.515$ and $D_{\infty}\sim1.737$, corresponding, respectively, to the sets where the measure is most rarefied and most concentrated. The fractal dimension of the set where all the singularities of the energy flux are located results to be close to space filling, being $D_0\sim2.96$. Finally the curve $f(\alpha)$ has unitary slope for $D_1\sim2.88$, which represents the dimension of the set (of volume zero) where all of the energy flux is concentrated asymptotically.

To conclude, using a simple cascade model for the energy flux towards the smaller length scales, we found, for the first time, the intermittency modification to the Kraichnan theory, thus showing the multifractal structure of MHD turbulence. Our model, which takes into account only an asymmetric energy flux, can be viewed as a first step in a correct understanding of multifractal structures in MHD [7]. However, the agreement between our results and the pioneering observations of intermittency in the interplanetary space fluctuations [3-5] allows us to explore further variants of the model, related to more accurate measurements of both standard and "Alfvénic" MHD turbulence [10]. A future paper will be devoted to get deeper insight in these problems.

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