Effects of External Noise on the Swift-Hohenberg Equation

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The Swift-Hohenberg equation is studied in the presence of a multiplicative noise. This stochastic equation could describe a situation in which a noise has been superimposed on the temperature gradient between the two plates of a Rayleigh-Bénard cell. A linear stability analysis and numerical simulations show that, in constrast to the additive-noise case, convective structures appear in a regime in which a deterministic analysis predicts a homogeneous solution.

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Hydrodynamic systems have been commonly used as prototypes to study instabilities out of equilibrium in spatially extended systems, from both an experimental and a theoretical point of view [1]. Recently, experimental results have focused on the effects of fluctuations in these systems. Examples in which noise seems to have an important role are the generation of convective rolls in Rayleigh-Bénard instabilities [2], the propagation of Taylor vortices in an unstable Couette-Taylor flow [3], electroconvection in nematic liquid crystals [4,5], and waves in film flow [6]. Noise is also important in the side-branching mechanism of solidification fronts [7–9]. In some of the experiments [2,7–9] the origin of the noise is not yet clear. Some theoretical studies of stochastic equations have been performed in order to explain such experiments [10–20]. However, the correct modeling of the experiments is far from being understood and the comparison between theoretical and experimental results is not successful in all situations [17,19]. Noise could have an internal or external origin, and it could appear in the dynamic equations or in the boundary conditions. The simplest way to consider fluctuations is by adding a thermal noise to the macroscopic equations. However, it has been observed that in some cases there is a large discrepancy between the values of the intensity required to achieve good agreement with experiments and those corresponding to a thermal noise assumption [17,19]. A theoretical study of different stochastic models and a comparison with experimental results seems to be a good procedure to clarify this situation.

The Swift-Hohenberg equation has been used to study Rayleigh-Bénard convection [1]. In the deterministic situation, for some value of the external control parameter, the fluid evolves from a homogeneous state to the generation of rolls. Fluctuations of internal origin are introduced by means of an additive noise [19–21]. The effects of deterministic perturbations, like temporal ramps [19] or sinusoids [14,19], on the external parameter have also been studied. In order to achieve better understanding of the effects of noise it is interesting to deliberately apply stochastic perturbations to the external parameter. In this Letter, we study a model which contains such fluctuations of external origin. This model could describe fluctuations in the temperature gradient that is externally applied to the system, giving rise to a new nonequilibrium situation. In this situation, the noise appears in the corresponding Langevin equation multiplying a function of the relevant variable. This is the so-called *multiplicative noise.* Its effects on the system are, in general, different from those induced by simple additive noise, because they depend on the state of the system [22,23]. In particular, a shift of the instability points has been observed in some experiments in liquid crystals [4,23].

We study a model which contains multiplicative noise and perform a linear analysis and numerical simulations to study the effects of this noise on the stability condition. In contrast to the additive-noise case, in which the effects of noise are only present for large noise intensities and consist on the appearance of disordered states, multiplicative noise induces a change in a linear stability analysis. The more surprising aspect is that this kind of noise induces the hydrodynamic transition to occur for a smaller value of the control parameter. That is, convective structures are predicted in a region near the deterministic point in which no pattern would exist without the external noise. This is also opposite to the result of the application of a sinusoidal perturbation, where the transition takes place for larger values of the control parameter [14,19].

To study the multiplicative-noise effects in pattern formation we consider the Swift-Hohenberg equation for the scalar variable $\psi(\mathbf{r},t)$ in two dimensions:

$$\frac{\partial\psi}{\partial t} = \left[\Gamma + \xi(\mathbf{r},t)\right]\psi - \left(1+\nabla^2\right)^2\psi - \psi^3 + \eta(\mathbf{r},t), \quad (1)$$

where we have nondimensionalized all the quantities. This equation is valid close to the onset and in the limit of infinite Prandtl number, so mean flow effects are ignored. The coefficient of the linear term is the control parameter, and $\xi(\mathbf{r},t)$ is an external noise which comes from assuming that this control parameter fluctuates around a mean value Γ. $\eta(\mathbf{r},t)$ is the internal noise. Both noises are Gaussian and white with zero mean and correlation $2D\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t')$. $D = D_I$ is the intensity of the

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internal noise and is proportional to the temperature, whereas $D = D_E$ is the intensity of the external noise, and is a second independent control parameter.

The transient and steady-state properties of this model are well known for the case $D_E = 0$. For a fixed small D_I and $\Gamma < \Gamma_c$, the system is in a homogeneous state with small random fluctuations induced by the additive noise. For $\Gamma > \Gamma_c$ there is a transition in which the system evolves from the homogeneous state to a convective ordered state composed of rolls. By increasing D_I this state becomes more disordered and could even disappear. The transition can be characterized by means of the structure function of the system $S(\mathbf{q},t) = \frac{1}{V}\langle\psi(\mathbf{q},t)\psi(-\mathbf{q},t)\rangle$, which is the Fourier transform in space of the two-point correlation function [20]. In order to study the effects of the multiplicative noise we analyze the behavior of the structure function corresponding to Eq. (1). The equation for the evolution of the structure function can be found by means of any standard stochastic method [24]. When calculating the average, the contribution of the additive internal noise does not change the deterministic stability condition of the linear analysis. However, due to the multiplicative character of the external noise, we find a new contribution coming from that term. In particular, in the equation for $\langle\psi_i\psi_j\rangle$ (where the subscripts denote a discrete lattice cell of size $\Delta x \times \Delta x$), the new contribution of the multiplicative noise $\langle\xi_i\psi_i\psi_j\rangle$ could be calculated by means of Novikov's theorem [25]:

$$\langle\xi_i\psi_i\psi_j\rangle = \frac{D_E}{\Delta x^2}\delta_{ij}\langle\psi_i\psi_j\rangle + \frac{D_E}{\Delta x^2}\langle\psi_i\psi_j\rangle. \tag{2}$$

In the linear approximation, the resulting equation for the structure function is

$$\frac{\partial}{\partial t}S(\mathbf{q},t) = 2\left[\Gamma + \widetilde{D_E} - \left(1 - 2q^2 + q^4\right)\right]S(\mathbf{q},t) + 2D_I + 2D_E\left(\frac{1}{2\pi}\right)^2\int S(\mathbf{q},t)\,d\mathbf{q}, \tag{3}$$

where $\widetilde{D_E} = \frac{D_E}{\Delta x^2}$ and we have considered $q\Delta x < 1$. The term proportional to $\widetilde{D_E}$, which contributes to the linear stability analysis, comes from the last term in Eq. (2). The integral term of Eq. (3), which couples all the Fourier modes, comes from the term of Eq. (2) which contains δ_{ij}. This term is always positive, so it will always have a destabilizing effect, if any.

Thus the presence of a multiplicative noise leads to the existence of an effective noise-dependent control parameter, $\Gamma + \widetilde{D_E}$. Therefore any perturbation of the homogeneous state will always grow if the condition $\Gamma + \widetilde{D_E} > \Gamma_c$ is obeyed. Hence, linear analysis predicts that the system under the presence of multiplicative noise will leave the homogeneous state in situations for which $\Gamma < \Gamma_c$. The nonlinearity will then stabilize the system in an ordered state.

We have also performed a numerical simulation of Eq. (1) in a two-dimensional lattice of 128×128 cells with a mesh size $\Delta x = 0.4870$. Integration in time was done by means of a standard Euler algorithm [26] with a time step $\Delta t = 1.7 \times 10^{-3}$. A finite-difference scheme was used to perform the integration in space [26]. Additive and multiplicative noise were introduced into the algorithm by means of a standard procedure [27]. The special values of the space and time steps used were imposed by a von Neumann stability analysis of the algorithm [13,26]. In Fig. 1, we present results for three different situations. Fixed boundary conditions have been used (field and normal derivatives are zero at the boundaries). In Fig. 1(a), we present the results for a case with $\Gamma = 0.37$ above its critical value (ordered state), and where only internal noise is considered ($D_E = 0$ and $D_I = 0.001$). As expected, the characteristic rolls are obtained. In Fig. 1(b) results for the same intensities of the noises ($D_E = 0$ and $D_I = 0.001$) but with $\Gamma = -0.05 < 0$ below the critical value are shown. No ordered pattern appears at this point. In Fig. 1(c), we present the results corresponding to the case in which multiplicative external noise is also

(a) (b) (c)

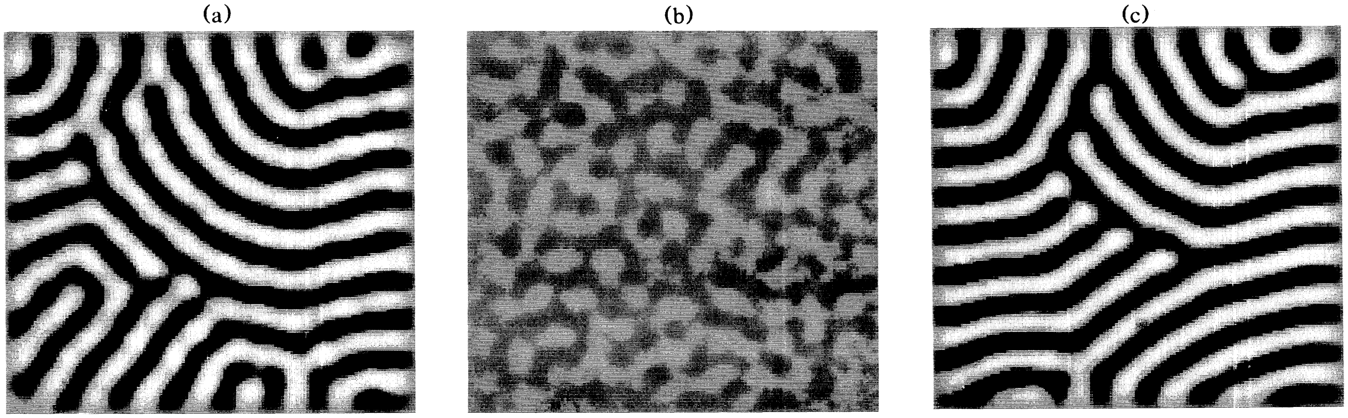

FIG. 1. Patterns corresponding to the same integration time (equal to 150 nondimensional time units) for the cases: (a) $\Gamma = 0.37$, $D_E = 0$, (b) $\Gamma = -0.05$, $D_E = 0$, (c) $\Gamma = -0.05$, $D_E = 0.1$. $D_I = 10^{-3}$ for all cases. The contrast is the same for the three figures.

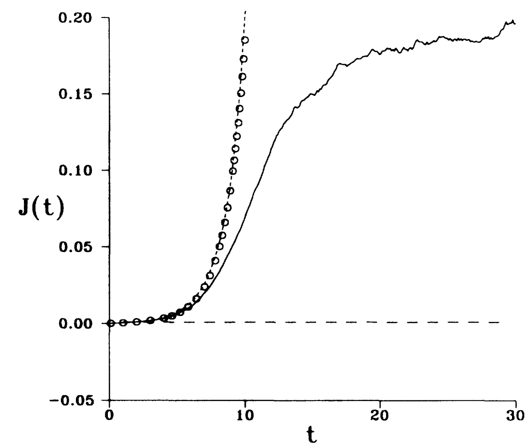

FIG. 2. Results for the transmitted flux versus time for a subcritical $\Gamma = -0.05$ with a small additive noise $D_I = 0.001$. The dashed line corresponds to $D_E = 0$ and the solid line to $D_E = 0.1$. The circles represent the numerical integration of Eq. (3) and the dotted line is the simulation of the linear version of Eq. (1), both of them for the case $D_E = 0.1$.

taken into account ($D_E = 0.1$ and $D_I = 0.001$). Γ has the same value as in Fig. 1(b), that is, below the critical deterministic value. In this case the characteristic pattern of the convective rolls also appears. This pattern is very similar to that induced by a supercritical value of Γ [Fig. 1(a)], but very different from the homogeneous pattern which appears for a subcritical value of Γ [Fig. 1(b)].

In Fig. 2 the perpendicular transmitted flux, which is defined as $J(t) = \frac{1}{V}\int \psi^2(\mathbf{r},t)d\mathbf{r}$ [2] and is proportional to the Nusselt number, is given as a function of time for the situations described in Figs. 1(b) and 1(c). Periodic boundary conditions are used now because they are required to compare with theoretical results, since in our analysis the Fourier lattice is periodic. The flux is zero for the homogeneous case of Fig. 1(b), and reaches a nonzero steady value for a not very large intensity of the external noise. In this figure, theoretical results obtained from a numerical integration of Eq. (3) are included. We obtain excellent quantitative agreement in the short-time regime. Furthermore, we also present results of the numerical simulations of Eq. (1) for the linear case ($\psi^3 = 0$), which agree with the linear theory in the complete temporal regime, as expected. In Fig. 3 the stationary flux J_{st} for a small intensity of the additive noise ($D_I = 0.001$) is shown for two different situations, with and without multiplicative noise. In both cases, the flux follows the same linear dependence against Γ but there is a clear positive shift proportional to the intensity of the multiplicative noise. We can thus obtain large enough values of the flux from a not very large D_E and a subcritical value of Γ. The threshold value of Γ given by the linear analysis for this case is approximately -0.17, in excellent agreement with the numerical simulations of

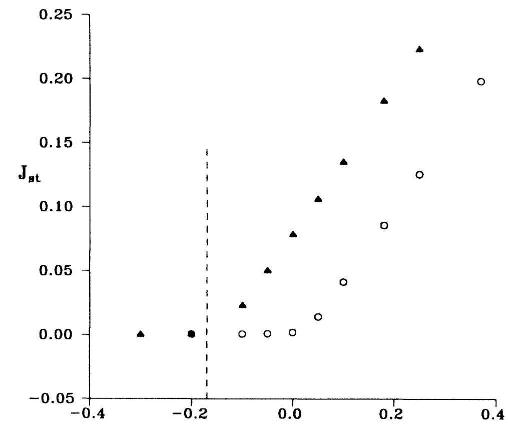

FIG. 3. Steady flux versus Γ for $D_E = 0$ (circles) and $D_E = 0.04$ (full triangles). The critical value of Γ predicted by the linear theory is indicated by a broken line.

Fig. 3 [28].

In Fig. 4, we present the results for the structure function calculated from our simulations for the three different cases of Fig. 1. For the two patterns of Figs. 1(a) and 1(c), we obtain almost the same structure function and both are very different from the homogeneous case of Fig. 1(b). The fact that both patterns are very similar is an interesting fact that requires to be studied in detail.

In this paper, we have discussed the effects of multiplicative noise in a spatially extended model. A complete description of these effects would require additional theoretical and numerical work. On the other hand, although there could be some practical difficulties to implement spatially extended multiplicative noise, experimental studies following the line opened by recent works on liquid crystals and other systems [23] would be of in-

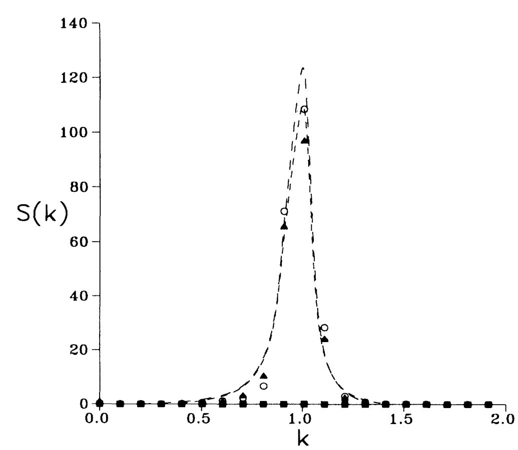

FIG. 4. Structure function at $t = 150$ for the three cases in Fig. 1: (a) circles, (b) squares, (c) triangles. Dashed lines are a guide for the eye.

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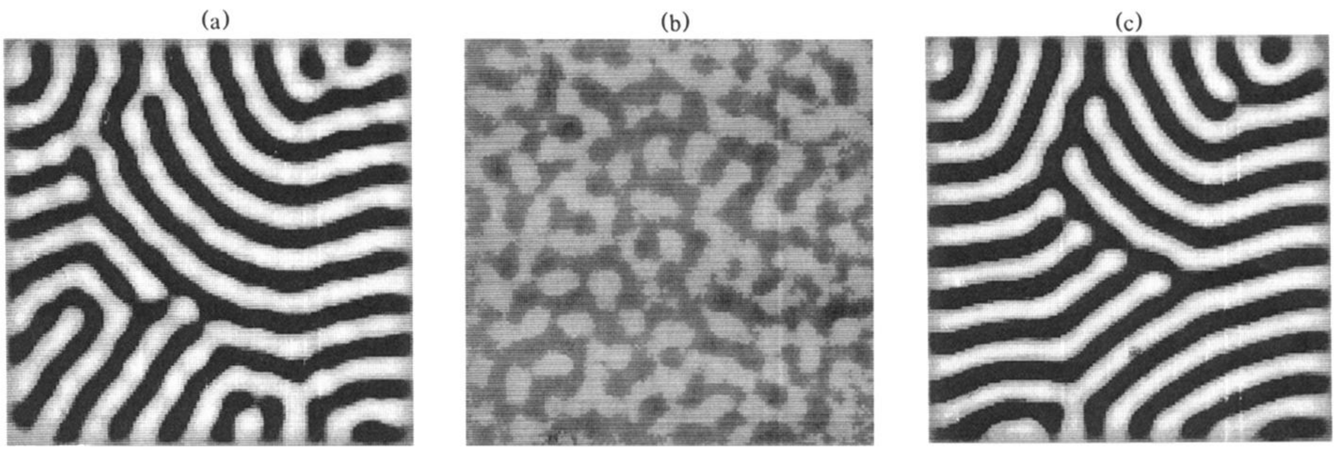

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