Nonlinearity of Pancharatnam's Topological Phase

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(Received 2 April 1993)

Since Berry introduced topological phases to describe the adiabatic transport of the spin of particles in parameter space, they have become a practical and descriptive concept for understanding interference phenomena in many quantum mechanical systems. For the photon two different applications of this concept were found in crystal optics: (1) the optical activity of helically wound optical fibers and (2) the development of the state of polarization on the Poincaré sphere. We present a remarkable optical "nonlinearity" of certain polarizing interferometers, which we discovered by using the concept of the topological phase on the Poincaré sphere and we demonstrate it with a simple experiment.

PACS numbers: 42.25.Hz, 42.65.−k

In optics the use of descriptive visual models has always played a leading role in the understanding of the basic concepts and their further developments. In crystal optics the Poincaré sphere is the most effective instrument for quickly acquiring detailed knowledge about the evolution of light in anisotropic media [1,2]. Simple trajectories on the surface of the Poincaré sphere describe the effect of optically active and birefringent plates or fibers on the state of polarization of light and give an illuminating insight into the function of polarization optic elements.

In 1956 Pancharatnam [3] investigated the effect of an analyzer P on the interference of two light beams with different states of polarization P_1 and P_2. He discovered that the analyzer introduces a phase difference γ between the beams, which only depends on the solid angle Ω (i.e., the spherical excess) of the triangle P_1P_2P on the Poincaré sphere and not on optical path, dispersion, and color; see Fig. 1:

$$\gamma = -\tfrac{1}{2}\,\Omega(P_1P_2P)\,. \qquad (1)$$

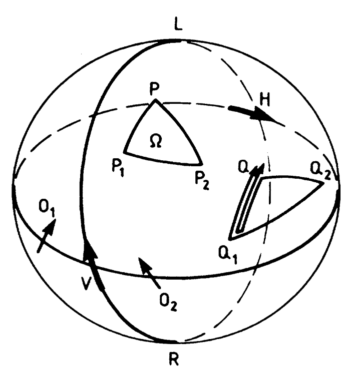

FIG. 1. Pancharatnam's theorem on the Poincaré sphere.

The idea that an analyzer introduces a phase difference between two light beams is more another way of looking at, rather than a new phenomenon in optics: Two plane waves (Fig. 2) are well known to interfere constructively or destructively after being resolved by a linear analyzer (after passing through the linear analyzer) in the V or H position. Using Pancharatnam's theorem, we would say that the spherical excess Ω of the triangle O_1O_2O (see Fig. 1) is zero for O in position V and 2π in position H (half of the surface of the sphere). Pancharatnam's phase γ is zero in the first case and $-\pi$ in the latter case. Thus, the interference is constructive and destructive, respectively.

Pancharatnam's discovery was remembered [4] when Berry [5] described his phase in 1984 and it was then incorporated into the theoretical framework of adiabatic rotations of spinors as a "geometrical" or "topological optical phase" [4,6–8]. With the methods of quantum mechanics a generalization of Pancharatnam's result was achieved. The dissipative nature of the resolving analyzer was found to be not a necessary condition and the relation between the topological and dynamical phases for general trajectories on the Poincaré sphere was cleared up. Equation (1) describes the geometrical phase difference (Pancharatnam's phase) between two parts of

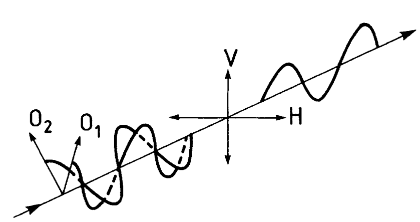

FIG. 2. A simple example of Pancharatnam's theorem.

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a coherent light beam, if one part is led around a closed loop of solid angle Ω on the Poincaré sphere.

This extension includes Pancharatnam's original statement: A coherent light beam of polarization Q_1 is split and one part is led around the closed loop $Q_1Q_2QQ_1$ of geodetic lines (great circles); see Fig. 1. This introduces a phase difference $-\frac{1}{2}\Omega$ between the two split beams at Q_1 which is conserved after their resolution by the analyzer Q.

An example such as the problem above can of course be analyzed with a simple picture like Fig. 2 or with other methods like the Jones matrix formalism, but an evaluation with Pancharatnam's phase is more descriptive and formally simpler in the general case of arbitrary states of polarization Q_1Q_2Q, where a perceptive superposition of waves is not possible at all.

Several experiments verified Eq. (1) [9–12]. In all cases a coherent light beam is split and one part is led around a variable closed path on the Poincaré sphere by means of crystal optic components which can move mechanically. The two split beams with their different history are then compared interferometrically. All trajectories on the Poincaré sphere are geodetic to avoid a dynamical phase change which may cover the wanted effect. Frequency modulation techniques [9,10] and the abandonment of the dissipative element (the analyzer P) carry these measurements far away from the plain arrangement of Pancharatnam's original idea and make their interpretation less transparent. The complicated interferometric equipment which has to be mechanically very stable has perhaps led to the view that Pancharatnam's phase is an effect which is only of academic interest. Only recently have interesting applications in interferometry and optical switching been proposed [13–15].

Here we want to show that the experimental verification and its interpretation can be very simple if one follows closely Pancharatnam's original idea. We measure an exotic phenomenon which we discovered by using the concept of Pancharatnam's phase and which is directly proportional to it. This effect is a nonlinear behavior of Pancharatnam's phase γ for certain interferometer arrangements which we will describe first. If two coherent light beams of different state of polarization P_1P_2 interfere after being resolved by an analyzer P to polarization P, we expect them to have accumulated a phase difference γ according to (1); see Fig. 3. P_i, P $(i=1,2)$ and the orthogonal state $\bar{P}_i$ lie on two great circles of the Poincaré sphere which enclose the spherical triangle P_1P_2P with excess Ω together with the equator. When P moves on the great circle $LHRV$ in direction L and H, the triangle Ω increases slowly if the angle $\Delta = \angle P_1P_2$ is small. But with P near H the two great circles $P_iP\bar{P}_i$ open up very fast and coincide with the equator when P reaches H. This means that Ω increases very fast in this region and consequently Pancharatnam's phase increases in a very strikingly nonlinear way. We measured this

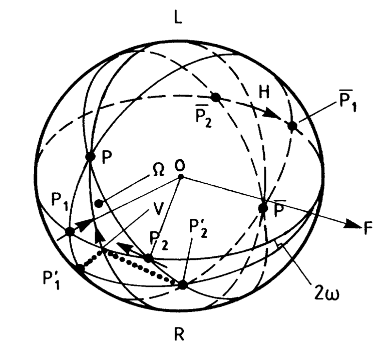

FIG. 3. Our experiment on the Poincaré sphere.

effect as a shift of the interference fringes of a two-beam interferometer. The fringe position depends only on the initial constant phase of the two beams and on Pancharatnam's phase $\gamma(\Omega)$ (1). With $\angle VP = 2\omega$, $\angle P_1P_2 = \Delta$, $\varphi = \angle P_1P = \angle P_2P = \arccos(\cos 2\omega \cos \Delta/2)$ spherical trigonometry leads to

$$\Omega = 2\arccos\left[\frac{\sin 2\omega \cos \Delta/2}{\sin\varphi}\right] + 2\arccos\left[\frac{\cos 2\omega \sin \Delta/2}{\sin\varphi}\right] - \pi . \tag{2}$$

Our experimental setup (see Fig. 4) deviates from this arrangement which leads to Eq. (2), but it is equivalent to it as we will show below. Two slits of 0, 25 mm diameter and 0, 5 mm distance serve as a two-beam interferometer. They are covered by two tiny pieces of the Polaroid sheet polarizer HN 22 with different orientation. The slits are illuminated by a vertically polarized HeNe-laser beam. We measure the lateral position of the interference fringes in the far field of the interferometer. Instead of moving the analyzer P up along $VLHR$ (see Fig. 3) we shift the position of P_1P_2 down with the help of a calibrated Babinet-Soleil compensator (45°, fast axis F) and resolve with a fixed analyzer in the V position; see the dotted lines in Fig. 3. The relevant triangle Ω is now $P'_1P'_2V$. The equivalence of this setup with our original one is shown as follows: A beam of polarization V is split; one part is led around the spherical triangle $VP_1P'_1$

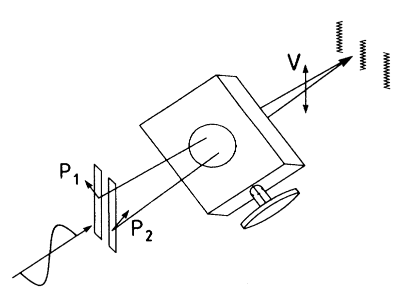

FIG. 4. Our experiment.

back to V, the other one around VP_2P_2' back to V in the opposite (negative) sense. The geometric phases gained by these round trips are given by $\gamma_{\text{geo}}(VP_1P_1') = -\frac{1}{2}\Omega_1$ and $\gamma_{\text{geo}}(VP_2'P_2) = +\frac{1}{2}\Omega_2$ with $\Omega_1 = \Omega_2 = \frac{1}{2}(4\omega\sin\Delta/2 - \Omega)$. The two arcs $\angle P_1'P_1$ and $\angle P_2'P_2$ are not geodetic lines on the Poincaré sphere and contribute with a dynamical phase. The dynamical phase gain corresponds to an additional rotation of the spinor which represents the state of polarization of the light on the Poincaré sphere around its eigenvector [7]. It is given by $\gamma_{\text{dyn}} = -\alpha/2\cos\beta$ where α is the angle of rotation—in our case 2ω—and β is the angle between the axis of rotation (F) and the eigenvector $\overrightarrow{OP_1'}$ or $\overrightarrow{OP_2'}$. We obtain $\gamma_{\text{dyn}}(P_1P_1') = +\omega\sin\Delta/2$ and $\gamma_{\text{dyn}}(P_2P_2') = -\omega\sin\Delta/2$. Adding up the contributions of the dynamical and geometrical phase gains for the two round trips and forming their difference one finds for the phase difference of the beams after their path $\gamma = \gamma_2 - \gamma_1 = -\frac{1}{2}\Omega$ as stated above.

Figure 5 shows the shift of four interference fringes with compensator position 2ω. The shift of the fringes is fast for 2 near $(n+\frac{1}{2})2\pi$ and slow for 2ω near $n2\pi$ $n=0,1,2,3,\ldots$ as expected from our qualitative estimation.

The intensity of the fringe pattern of two beams of intensity J_1 and J_2 with polarization P_1 and P_2, respectively, after being resolved by the analyzer P to the same state of polarization is given by [2,3]

$$J = J_1\cos^2 b + J_2\cos^2 a + 2\sqrt{J_1J_2}\cos a\cos b\cos(\delta+\gamma)\,, \tag{3}$$

with $2a = \angle P_1P$, $2b = \angle P_2P$. In our symmetrical case with $J_1 = J_2 = \frac{1}{2}J_0$ and $\cos 2a = \cos 2b = \cos 2\omega\cos\Delta/2$ we obtain

$$J = J_0\cos^2\left(\frac{\delta+\gamma}{2}\right)(1+\cos 2\omega\cos\Delta/2)\,. \tag{4}$$

This is a completely modulated intensity function with a fringe contrast $V=1$ in the whole parameter space. $\delta+\gamma$ describes the lateral fringe position.

Figure 6 shows a measurement of the shift of one fringe in comparison with the theoretical curve (1) (2). The agreement is very good. The "nonlinear" behavior of Pancharatnam's phase increases when $\Delta = \angle P_1P_2$ decreases, i.e., when the states of polarization P_1 and P_2 lie close together on the Poincaré sphere. Very abrupt phase changes can be obtained but one pays with a rapid loss of intensity of the fringe pattern in the instability region around $2 \approx \pi$. The fringe intensity $J(2=\pi)$ goes down with $1-\cos\Delta/2$; see Eq. (4).

Bhandari [14] has described and measured the unbounded nature of Pancharatnam's phase and Martinelli and Vavassori [13] have explained its usefulness for endless polarization control. The continuous shift of the fringes over several periods 2π in Fig. 5 is not unbounded in our experiment since the shift of polarization P_1, P_2 with the compensator cannot be endless. A simple change of the experiment using a rotating linear analyzer for P and two light beams with the same small ellipticity but of different sign shows the same fringe behavior. The continuous nonlinear shift of the interference fringes with rotating analyzer P can directly be seen in an ocular.

The nonlinear behavior of Pancharatnam's phase in special interferometers can be used to switch light faster and with less retardation between the beams. In our case a phase amplification of 1 order of magnitude between the compensator and the two slit interferometer occurs. Also a "phase rectification" is possible if the working point of the compensator is shifted to the bending region of the characteristic curve (Fig. 4). These effects can be applied for precise measurements of polarizer orientations.

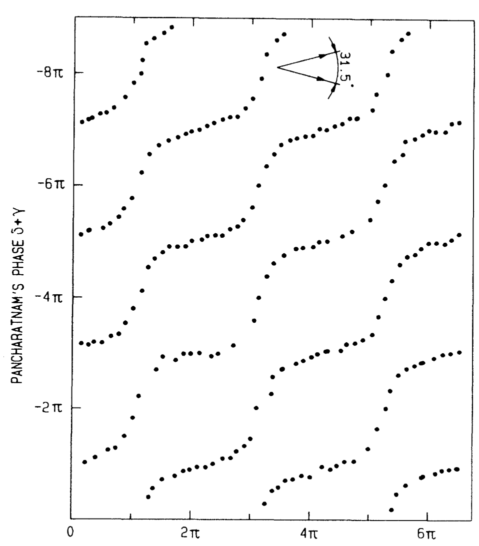

FIG. 5. The shift of interference fringes according to Pancharatnam's theorem.

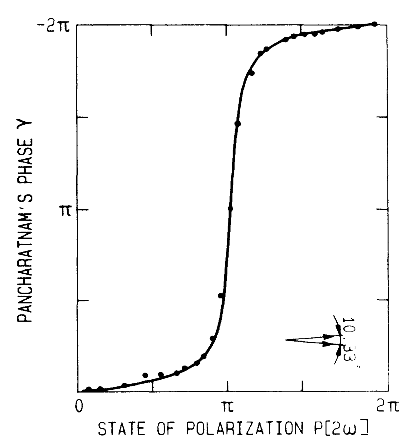

FIG. 6. Comparison of our experiment (dots) and Pancharatnam's theorem (line).

Pancharatnam's phase depends only on the area Ω on the Poincaré sphere and is in principle achromatic. This can be useful for applications in optical communication where interferometric devices switch many different spectral channels. In an interferometer which is scanned with this method all the light beams pass through the same variable element—the analyzer P—, so their spatial separation can be very small. The phase modulation amplitude of a polarizing hologram can be actively varied in this way.

We want to emphasize that Pancharatnam's phase is an exciting concept in optical education, helping the student and the experienced practitioner to acquire a better understanding. It shows the superiority of group theoretical over arithmetical methods and will hopefully inspire the development of new devices and strategies in optics. The discussion of the nature of topological phases in optics [16] shows that a simple crystal optic effect has deep roots down to a level where our basic understanding or better misunderstanding of physics is demanded.

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