Slow Dynamics from Noise Adaptation

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We discuss a new mechanism generating long range temporal correlations in dynamical systems coupled to a source of white noise. The external noise induces dynamical events uncorrelated on a *logarithmic* time scale and produces a fluctuating output with "$1/f$" power spectrum. This behavior requires a complex phase space with many traps, which can arise due to strong cooperative effects. As a demonstration, we numerically analyze a system of many coupled degrees of freedom, which is externally driven and subject to a white noise perturbation. We find both Poisson statistics for events in log(time) and the $1/f$ power spectrum.

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Little is known theoretically about driven dissipative systems with very many fixed points. Studying an example in detail can thus provide significant insights on how phase-space properties are reflected in the relaxation behavior. In this Letter we show for a particular model that white-noise-induced hopping events between attractors only decorrelate on a *logarithmic* time scale. Any variable changing with the events is then slow relaxing and has a (nonstationary) $1/f$ power spectrum [1], independently of tuning parameters. The condition is that attractors of all "depths" exist close to any point in phase space. Since the coarse-grained dynamics is then determined by the noise history, we call the mechanism *noise adaptation.*

The mechanism is demonstrated by extensive simulations of a large automaton, mathematically akin to the mappings introduced by Coppersmith and Littlewood [2] to study pulse-duration memory effects in charge-density waves (CDW), and used by Tang *et al.* [3] to discuss the concept of phase organization.

Our model is meant to describe phase slips in CDW, a possible source for the low frequency broadband noise experimentally found [4, 5] but not reproduced by the standard Fukuyama-Lee-Rice model [6, 7]. We believe that our results should also apply to the statistics of jumps between mode-locked states in ac driven CDW [8]. However, we here defer a discussion of CDW specifics, focusing on general aspects.

In the sequel, we first define the automaton. Second, we show analytically how its—idealized—features lead to nonstationary $1/f$ noise. Third, we describe the numerical verification of these features. Finally, we compare our mechanism to other models with a slow dynamics.

We define variables σ_i on a square lattice (size 125×125 was used in most simulations), which represent coarse-grained phase *velocities* of the CDW, rather than phases. The dynamics includes (1) the tendency of nonlinear coupled oscillators to lock into each other, i.e., move with the same (phase) velocity; (2) a competing random spatial variation of the pinning forces; and (3) a viscous force, proportional to the velocity gradient. A spatial variation of σ indicates the presence of phase slips [9].

In the update rule we first calculate the force at site i as $f_i = \max[C(\nabla^2\sigma)_i + E - p_i + n_i(t), 0]$, where E is the driving field, p_i is the random positive pinning force, $n_i(t) \ll E$ is the noise perturbation—which can be zero—at iteration t, C is a numerical coefficient, and ∇^2 is the lattice Laplacian. The velocity field is then asyncronously updated by $\sigma_i \leftarrow \mathrm{NINT}(\sigma_i/2 + f_i/2)$, where the NINT (nearest integer) function mimics the locking and the inertial term prevents blinking.

Adding noise induces hopping between the many attractors or "traps" of the noiseless dynamics. In practice, no velocity changes are observed until a fluctuation in the force occurs which is large enough to induce an avalanche. This is a rearrangement of the σ_i's which is both temporally and spatially localized, and which we treat as effectively instantaneous.

We define the "depth" of an attractor as the smallest amplitude of *bounded* noise, which will eventually move the system out of the attractor basin. Operationally, the depth is calculated by (1) evolving for sufficiently long time from a trap under the influence of white noise, which is uniformly distributed in the finite interval $(0, N_{\max})$, (2) removing the noise, whereby the system again relaxes to an attractor, and (3) checking if the initial and final configurations differ and using the information to modify $N_{\max}$ by bisection.

The scale for the noise magnitude and trap depth is fixed by the NINT function, which models the locking between different degrees of freedom. A perturbation of size 1 is considered large, as it always induces a change in the next time step.

Additive noise of bounded magnitude eventually leads the system into a trap it cannot escape. Let us instead consider the more interesting case where spikes of any size can occur, with probability decreasing with size. As expected [2, 3], we find that the system initially explores shallow attractors. Let us then assume that, at some early time, it sits in a trap of depth $d_1 \ll 1$. At a later time a noise kick of intensity $I_2 > d_1$ triggers an avalanche, eventually leading to a new trap of depth d_2. Our central observation is that the depth of the *new* attractor is very close to the magnitude of the *previous* noise spike causing the avalanche, i.e., $d_2 \sim I_2$. Therefore, the next avalanche can only be triggered by a spike of magnitude $I_3 > I_2$. We call this mechanism *noise adaptation* since the distribution of traps available to the system adapts to the noise. Physically, this hypothesis is a generalization of the marginal stability hypothesis earlier advanced [2]. The generic state of the system is marginally stable to the variation of driving parameters as well as to the history of external noise.

Since avalanches are confined, the memory of a noise spike only extends to some finite region. Within each of these, the avalanche statistics is similar to that of records among the noise spikes, where a record is a spike larger than all the previous ones. As shown later, the probability $P_n(t)$ of having n records in a fixed time span $t > n$ is asymptotically given by

$$P_n(t) = \frac{1}{t}\frac{(\ln t)^{n-1}}{(n-1)!} \quad t \gg n \gg 1 \tag{1}$$

independently of the noise distribution. Equation (1) is a Poisson distribution in $\ln t$. Hence, the quantities $\Delta_k = \ln t_k - \ln t_{k-1}$, where t_k is the time of the k'th event [10], are independent stochastic variables with the common cumulative distribution

$$P(\Delta) = 1 - \exp(-\Delta) . \tag{2}$$

Time correlations between events disappear when time is measured on a *logarithmic* scale.

In this analogy, the number of avalanches happening in time $\ln t$ in a system of size S should be a sum of N_d Poisson variables, where S/N_d is related to the characteristic size of the avalanche. The sum is itself Poisson distributed with average $N_d \ln t$, and the corresponding average value of Δ is $1/N_d$. The number of "domains" N_d must scale with the system size. All these relations are confirmed by the simulations.

Let $\tau = \ln t$ and consider a mesoscopic variable $v(\tau)$ changing due to events uncorrelated on a logarithmic time scale. Its distribution will evolve according to a master equation defined by some negative definite operator $\mathcal{L}$ acting on (τ, v) [rather than (t, v)] variables. If $\mathcal{L}$ has a point spectrum and λ_0 is the smallest eigenvalue, for τ of the order of $1/\lambda_0$ the autocorrelation is $C_v(\tau) \simeq \exp -\lambda_0\tau = t^{-\lambda_0}$. The power spectrum is $S(f) = 1/f^{1-\lambda_0}$. In the limit $\lambda_0 \to 0$ one finds $1/f$ noise.

Under nonstationary conditions such as ours, any initial condition i has generically a component in the eigenspace of λ_0, and therefore decays asymptotically proportionate to the autocorrelation. Hence $S(f) \simeq \int \exp(-i\omega t)\langle v_i(t)\rangle$, where $\langle v_i(t)\rangle$ is the ensemble average of v for an initial condition sharp at i. It can be estimated as a moving average over a sufficiently large time window of a single trajectory.

The extent to which the strong nonstationarity of our mechanism would appear in spatially averaged data depends on the magnitude of the relative change in v per event. In our case the change is very small, typically of the order 10^{-3} for several time decades of simulation.

We now show how Eq. (1) follows from the observation that the depth of a new trap is equal to the instantaneous magnitude of the preceding noise spike. Consider a discrete time sequence of events (the sizes of the noise spikes), described by independent identically distributed positive stochastic variables N_i, $i = 1, \ldots, t$. A record E_i occurs at time i if and only if $N_i > \max(N_1 \cdots N_{i-1})$. We want to derive the probability $P_n(t)$ that $0 < n \leq t$ records will occur in a fixed time span t.

For all $n > t$ one simply has $P_n = 0$. The probability of one and only one record in time t is $P_1(t) = 1/t$ (this is just the probability that the first out of t choices is the largest). By the same token, the probability of a record at time m is $p_m = 1/m$. The first record always happens at time 1. The probability of exactly two records in time span t, conditional to the second one occurring at m, is

$$P_{2|1,m}(t) = \frac{\prod_{k=2}^{t}(1-1/k)}{m-1} = \frac{1}{t(m-1)} . \tag{3}$$

Summing the above over all possible m values, we obtain

$$P_2(t) = \sum_{m=2}^{t} \frac{1}{t(m-1)} \simeq \ln(t)/t . \tag{4}$$

For $k+1$ events occurring at times $1 < m_1 < \cdots < m_k <= t$, we similarly get

$$P_{k|1,m_1,\ldots,m_{k-1}}(t) = \frac{1}{t(m_1-1)(m_2-1)\cdots(m_{k-1}-1)} . \tag{5}$$

Taking $q_k = m_k - 1$ we therefore find, by summing over the indices,

$$P_k(t) = \frac{1}{t}\sum_{q_1=1}^{t-k+1}\frac{1}{q_1}\cdots\sum_{q_{k-1}=q_{k-2}+1}^{t-1}\frac{1}{q_{k-1}}. \tag{6}$$

Approximating the sum with integrals, we regain Eq. (1) in the limit $t \gg k \gg 1$.

We can rephrase this in terms of single particles trapped in local wells and kicked by noise with an unbounded distribution $f(x)$. Let $P(y,t)$ be the depth distribution of wells occupied at time t. A particle escapes if a noise spike exceeds the depth of its well. It then falls into a new well of depth equal to the size of the spike. The master equation for P is

$$\frac{\partial P(y,t)}{\partial t} = -P(y,t)F(y) + f(y)\int_0^y dx\, P(x,t) , \tag{7}$$

where $F(y) = \int_y^\infty dx\, f(x)$. Making the substitution $P = (f/F)g(F,t)$, one can readily show that asymptotically $[t \to \infty,\ Ft \sim O(1)]$ the solution of Eq. (7) is that g be a *homogeneous* function of Ft. Then the total event rate is

TABLE I. The sample average and standard deviation for the attractor depth as a function of the noise amplitude, with $E = 6$. The statistics are based on 50 runs with different random initial conditions and noise histories.

Amplitude	E(Depth)	σ(Depth)
0.006	0.0065	8.19×10^{-4}
0.01	0.0098	3.53×10^{-4}
0.05	0.049	3.19×10^{-3}
0.10	0.098	5.20×10^{-3}
0.15	0.146	3.19×10^{-3}
0.20	0.186	5.73×10^{-3}
0.30	0.284	1.15×10^{-2}
0.40	0.372	2.15×10^{-2}
0.50	0.469	1.91×10^{-2}

$$\frac{dn}{dt} = \int_0^\infty dx P(x,t)F(x) = -\int dF\, g(F,t) \sim 1/t\,. \tag{8}$$

The average number of events then grows as $\ln(t)$.

We present results for $C = 0.4$ and for pinning forces which are zero on a random sublattice comprising half of the sites and otherwise drawn from an exponential distribution with average 7. The field values explored range from $E = 2$ to $E = 100$, with most simulations done for $E = 6$ and 10. We utilized different types of unbounded white noise. For convenience we mainly used an exponential amplitude distribution, randomly changing the sign of the deviate (with equiprobable signs). We also varied the *spatial* noise correlations, considering (a) same noise on each site, (b) uncorrelated magnitude but equal sign, and (c) both magnitude and sign uncorrelated. Also, we used Gaussian spatially and temporally uncorrelated noise, generated by the Box-Müller algorithm [11]. In all cases the "amplitude" of the noise is half the standard deviation. No important differences arise from varying noise parameters.

The model was simulated on a 16×2^{10} processor MasPar parallel computer, with a total numerical effort of the order of 10^3 h.

Starting the noiseless dynamics from different random initial configurations, we observed that for a wide range of E values (6–20) each attempt would lead to a different trap, implying the existence of a large number of attractors.

Table I shows that the depth of a trap is almost identical to the amplitude of the noise which put the system into it. To verify Eqs. (1) and (2) we performed, for several sets of model parameter values, 50 different simulations. Each had a different external noise history, ran over either 10^5 or 2×10^5 time steps, and comprised about 100 events. For each parameter set, the empirical quantiles of the interevent intervals Δ_k scaled by their means are compared with the predicted exponential distribution with average 1. As shown by Fig. 1(a), the agreement is good. Figure 1(b) shows, for a single parameter set, the average number of events in the intervals $(0, t_i)$ for a selected number of t_i's. Since the number of domains N_d is the slope of the curves, it is apparent that N_d scales with the system size. Finally, we show in Fig. 2 $\log_{10} S(f)$ vs

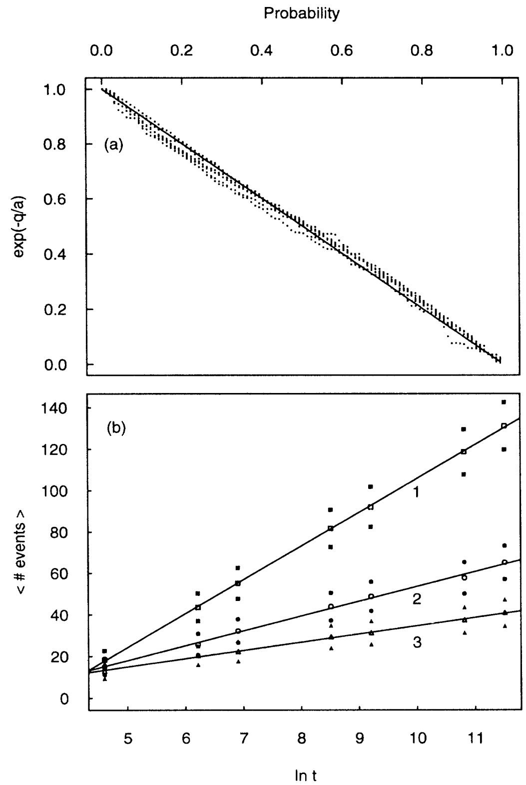

FIG. 1. (a) The empirical quantiles q of exponentially distributed "waiting times" $\Delta_k = \ln t_k - \ln t_{k-1}$ with average a obey $q \simeq a \ln(1-P)$, for $0 < P < 1$. To check this relation we use 8 different sets of Δs, each of approximately 2500 points, and each corresponding to a different parameter set. For each set, we extract 100 equally spaced empirical quantiles, and estimate a by the arithmetic mean of the Δs. In (a) $\exp -q/a$ vs P is plotted as a dot, while the line describes the exponential distribution. (b) (a) implies that the number of events happening in $(0,t)$ is Poisson distributed with average $N_d \ln t$, where N_d is the number of domains in the sample. To demonstrate that N_d scales with the system area, we plot the estimated averages vs $\ln t$ for sizes 1: 125×125, 2: 89×89, and 3: 63×63. The slopes of the corresponding linear fits rounded to the nearest integer are 16, 7, and 4, which is almost correct. Since $N_d = 1/a$, with a defined above, it can also be estimated from the distribution of the Δs. This yields 13, 7, and 4, respectively. The disagreement might be due to an undercounting of spatially separated but temporally overlapping cascades. The averages were estimated as means over 50 data sets, with different noise histories. The driving field is $E = 10$; the noise magnitude is exponentially distributed with standard deviation = 0.005. The vertical distance between the filled symbols on each side of the averages is twice the standard deviation of the *distribution*, i.e., approximately $\sqrt{50}$ times *larger* than the statistical error on the central point.

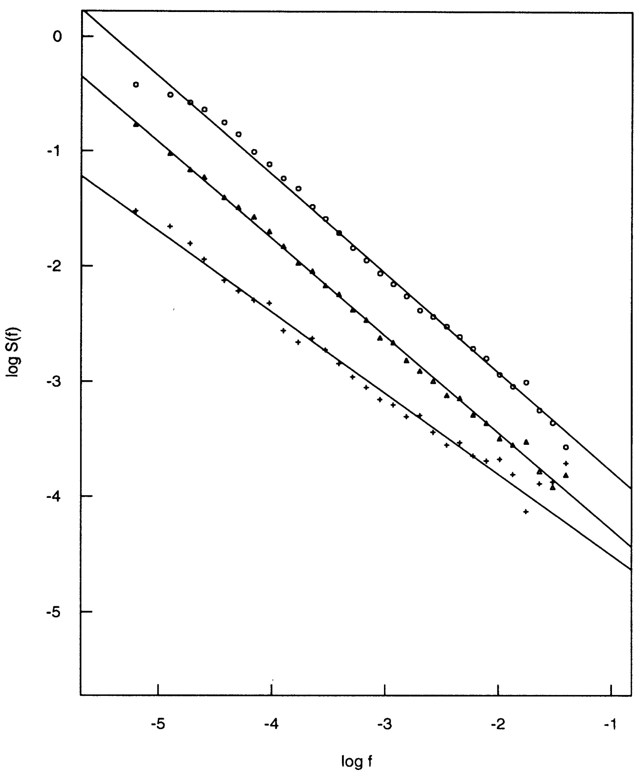

FIG. 2. The figure shows $\log_{10} S(f)$ vs $\log_{10} f$. As discussed in the text, under nonstationary conditions $S(f)$ is approximately the Fourier transform of the spatially averaged velocity $v(f)$, again averaged over (5) different noise histories. The parameters are, from top to bottom, $E = 6$ and noise amplitude = 0.005; $E = 6$ and noise amplitude = 0.001; and $E = 10$ and noise amplitude = 0.003. The spacing between the plots is artificially enhanced by parallel shifts. The slopes of the lines are 0.86, 0.84, and 0.70, respectively. The initial condition is a trap generated from a random configuration by a noiseless dynamics, with the same field as subsequently used in the noisy evolution. All Fourier transforms use 10^6 time steps.

$\log_{10} f$ for three different parameter sets, each obtained by averaging over 5 different traces of 10^6 time steps. The spectra have $1/f$ character through 5 decades, with an exponent weakly depending on the parameters, as previously discussed.

Slowly decaying correlations can arise in different ways, e.g., (i) by a disorder average of the correlation of thermally relaxing two-level systems [1, 12], (ii) in hierarchical models of thermal relaxation [13], and (iii) in open driven dissipative systems, as our case, which models a CDW system. Both (ii) and (iii) feature multiple attractors; however, in case (ii) a large energy change requires a large configuration change, while in (iii) attractors of all sizes are available within few dynamical steps. This is more similar to biological evolution as modeled by walks on a *rugged* high-dimensional fitness landscape [14]. The difference shows up clearly in the data: while the hierarchical diffusion exponents are temperature dependent, our exponents are independent of all *noise* parameters.

Finally, in contrast to the model of self-organized criticality (SOC) [15], our avalanches are temporally and spatially confined. Our picture is much closer to models of CDW below threshold (i.e., away from the critical point), having very many pinned states, related to each other by local rearrangements [6, 16]. The noise-induced logarithmic relaxation of the angle of repose of a sandpile [17] may provide one physical realization of our model.

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