Vortex Reconnection in Superfluid Helium

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A useful physical model for superfluid turbulence considers the flow to consist of a dense tangle of vortex lines which evolve and interact. It has been suggested that these vortex lines can dynamically reconnect upon close approach. Here, we consider the nonlinear Schrödinger equation model of superfluid quantum mechanics, and use numerical simulation to study this topology changing core-scale process. Our results support the idea that vortex reconnection will occur whenever filaments come within a few core lengths of one another.

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Ever since the original work of Feynman [1], the idea of treating superfluid turbulence as due to a dense tangle of (quantized) vortex lines [2] has been very attractive. The organized pattern of vortices which appear, for example, during the rotation of the superfluid can become a chaotic web of interconnected lines, greatly increasing the flow resistance. It remains a challenge, however, to characterize this state in a manner which would enable the calculation of flow behavior as a function of system parameters.

In a set of pioneering papers, Schwarz [3] introduced a numerical simulation technique to address this issue quantitatively. The hydrodynamics of liquid helium is reduced to the tracking of a set of vortex lines which evolve under the locally induced flow [4], as well as a frictional force due to interactions with the normal fluid. In addition, he assumed that nonlocal terms which become important as a piece of vortex filament closely approaches another such piece (or the system boundary) would lead to vortex reconnection. In fact, this reconnection mechanism is essential in sustaining the vortex tangle state. The success of this approach in explaining experimental findings has been reviewed recently by Donnelly [5].

The purpose of this paper is to provide a detailed calculation of vortex reconnection events, lending support to the aforementioned *ad hoc* assumptions. To do so requires a *microscopic* quantum mechanical model of superfluid helium, since it is on the core length scale that the relevant dynamics takes place. The simplest such model is the nonlinear Schrödinger equation for the boson wave function [6,7]

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+(v_0|\Psi|^2+w)\Psi\,, \qquad (1)$$

where the nonlinear term arises due to the boson-boson repulsion. In this language, a vortex is a two-dimensional solution of this equation of the form $\Psi_0=f(r)e^{i\phi}$, in cylindrical coordinates, where $f\to 0$ as $r\to 0$ and $f\to 1$ as $r\to\infty$ [8]. The superfluid velocity is given by $\mathbf{v}=\hbar f\times m^{-1}\nabla\phi$, which upon integration at large r leads to the quantized circulation $\Gamma\equiv\oint d\mathbf{l}\cdot\mathbf{v}=2\pi\hbar/m$. Needless to say, this model does not accurately represent all of the relevant quantum mechanics that could be taking place in a vortex core; nevertheless, we argue that the basic regularization needed for the recombination process is indeed present in Eq. (1), and therefore our results capture the true dynamics of this system.

Before proceeding to the details of our calculations, it is worth mentioning other studies of vortex reconnection. Within classical fluid mechanics, much effort has gone into understanding the role of viscosity in the reconnection process [9–11]. There has also been work on the time-dependent Landau-Ginzburg equation [12]. In both of these cases dissipative effects are all important; in contrast, our system is Hamiltonian. Somewhat closer to our situation is that afforded by the evolution of cosmic strings via the nonlinear Klein-Gordon equation [13]; those systems are Hamiltonian but the dynamics is fully relativistic, unlike the situation relevant for liquid helium. Finally, it is possible to directly observe vortex reconnection by using defect lines in liquid crystals [14].

We proceed by first removing the linear term in w by a global phase shift, $\Psi\to e^{-iwt/\hbar}\Psi$, and then rescaling time, space, and the modulus of Ψ to set the remaining constants in (1) to unity. In these units, the amplitude f is within a few percent of unity at a distance $r\sim 5$ from the core location [5]. An initial vortex filament configuration is determined by a core location and a fixed vorticity direction; the field in the plane perpendicular to the vorticity vector is just the field of a single vortex with the given core position; for example, a $\hat{z}$ vortex with a core at (x_0,y_0) gives $\Psi(x,y,z)=\Psi_0(x-x_0,y-y_0)$. The complete initial configuration is given by *multiplying* each of the single vortex fields for the filaments present in the computational volume. In what follows, we address the case of two nearby vortex filaments, varying the relative angle between the vorticity vectors.

To numerically integrate the nonlinear equation of motion, we use a split-step spectral method [15]. Each time step is split into two segments, the first of which in-

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tegrates the nonlinear term in real space, and in the second the Laplacian operator is done in Fourier space. In each segment the term in question is local, and in fact the integration amounts to multiplication by an infinitesimal phase, insuring the conservation of the normalization, $\int|\Psi|^2$, and of the energy.

The boundary conditions require some care. We use a periodic box for the physical reason that the vortex cores must extend to the ends of the system, and any explicit constraint on their behavior there might prejudice their motion. Furthermore, periodicity allows us to use fast-Fourier-transform routines, leading to a very efficient parallel computation. Now it is clear that the nontrivial phase $e^{i\phi}$ of the single vortex configuration (for any plane perpendicular to the vorticity) would wreak havoc with any direct attempt to impose periodic boundary conditions for our primary box. To resolve this difficulty, we extend our computation to the region $-L<x,y,z<L$; furthermore, we replace each single vortex line with a set of four parallel lines of alternating vorticity at positions arrived at via reflection through the coordinate axes. For the example of our vortex at (x_0,y_0), we would add antivortices at $(-x_0,y_0)$ and $(x_0,-y_0)$ and a vortex at $(-x_0,-y_0)$. Once this is done, the wave-function phase is a well-behaved function and the spectral part of the algorithm can be implemented without difficulty. Of course, one must take a large enough box and place the vortex filaments with sufficient care such that the relevant interactions are not overwhelmed by unwanted boundary effects.

According to Schwarz [3], when two vortices approach, the long-range interactions tend to drive the cores together so to be antiparallel (we will refer to this as 180°) at the point of closest approach with some oscillation along the core. In Fig. 1(a), we show an initial configuration consisting of a vortex-antivortex pair with sinusoidally varying core coordinates. In this and other figures we plot the surface $|\Psi|^2=0.3$, to indicate the location and distortions of the cores. The simulation grid is 64^3, the half-box size is $L=20$, and the average separation of the cores is 4 units. A mirror-image pair is present in the lower half-space, 24 units away, but not displayed. As the simulation progresses the cores merge at the point of closest approach [Fig. 1(b)], reconnect [Fig. 1(c)], and then retract from each other [Fig. 1(d)]. The initial and final stages are dominated by the long distance interaction already present in a vortex simulation [3]: The antiparallel cores move downwards in tandem. However, the intermediate reconnection stage is dominated by the core dynamics and has not been previously studied. The numerics have been checked by halving the time step, or by doubling the resolution to 128^3 while retaining the box size and core separations. In both cases no discernable change is found. We have also checked that the results are insensitive to the periodic images by a 128^3 simulation with twice the box size and 48 units separation from the image below. In this case the core shapes differ in

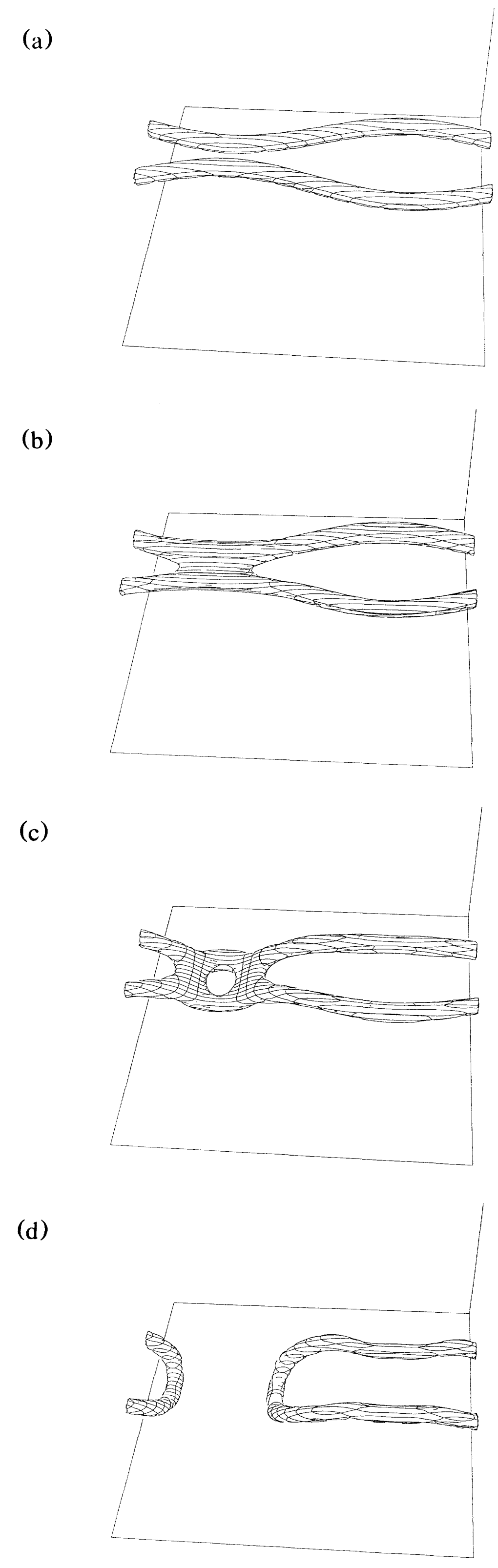

FIG. 1. Reconnection of antiparallel vortices: (a) initial configuration, (b) time 3.0, (c) time 10.0, and (d) time 20.

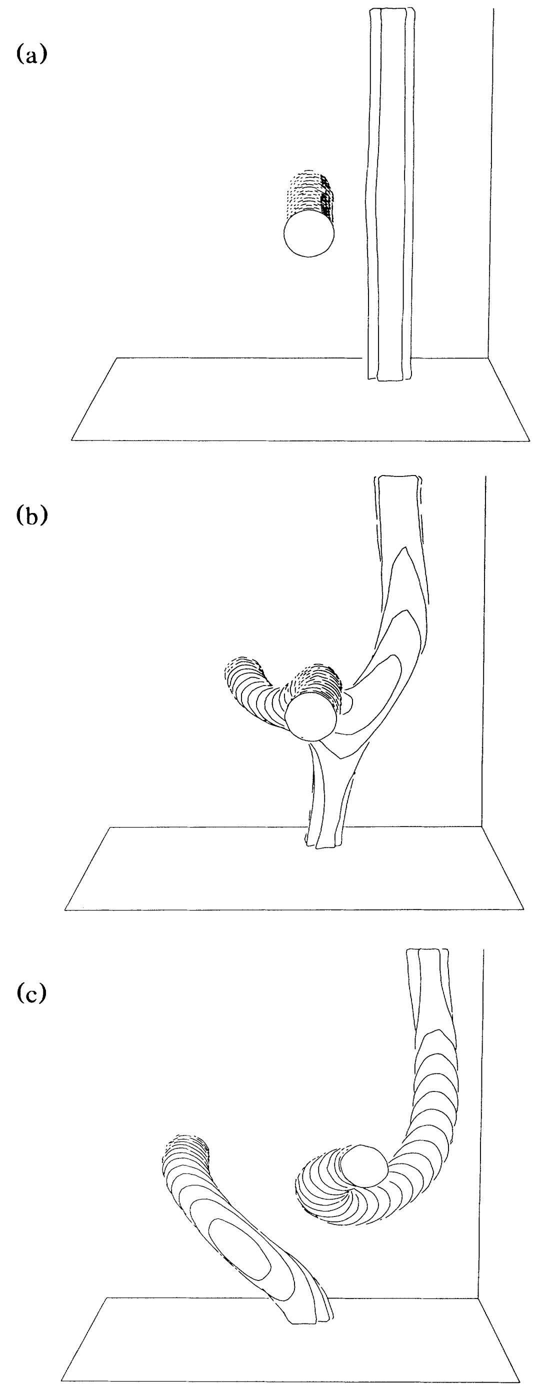

FIG. 2. Reconnection of 90° vortices: (a) initially, (b) time 7.0, and (c) time 20.

fine detail but reconnection persists. Other runs with different initial curvature and equal or closer initial separation always reconnect; if the initial separation is too large, the cores simply translate. Thus the long-range interactions are required to bring the vortices close enough together for quantum mechanics to act and cause reconnection.

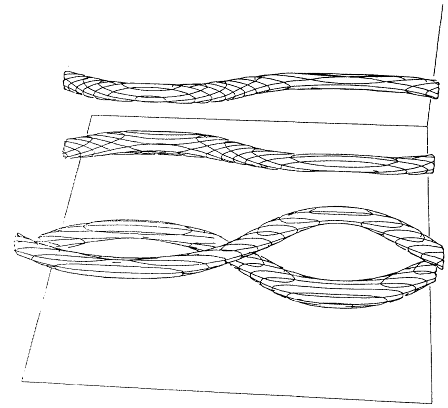

FIG. 3. Parallel vortice avoiding reconnection: time 20.0 after an initial configuration similar to Fig. 1.

Although the 180° configuration is the relevant one for superfluid turbulence, it is of interest to consider other orientations. In Fig. 2(a), we see an initial vortex line pair with 90° relative orientation. Again, the cores merge at the point of closest approach, Fig. 2(b), and the lines eventually reconnect, leading to the configuration shown in Fig. 2(c). Again, the fact of reconnection is insensitive to our lattice size and to the box size, as long as the interacting pair are much closer to each other than to any boundary of the primary computational box, although detailed shapes vary with the distance to the image vortices.

If the vortices are initially parallel (0°), however, the behavior differs. At relatively large initial separation, the vortices either rotate about their common center or slide past each other (Fig. 3), while in initially closer cases the cores may overlap for a time, then split, merge again, and so on, without any reconnection. In this case the image vortices and the periodicity act to keep the pair in proximity. We have also studied cases where the initial orientation is 45° and 135°; the former resembles the parallel case, with no reconnection, while the latter case reconnects in a manner similar to Figs. 1 and 2. The constraint of periodicity does not lend itself readily to other initial orientations, and we have not considered them.

One may ask how the reconnection process can be consistent with Kelvin's theorem, which states that the circulation around a closed contour moving with the fluid is conserved in the absence of viscosity. If we consider a contour C circling one vortex in Fig. 1(a), then unless C is precisely at the (symmetric) point of closest approach, it is carried away from the merger region by the flow and continues to circle the vortex after reconnection. If instead C is precisely at the symmetric point it is not advected away, but instead it is trapped in the merger region. While the nonlinear Schrödinger equation is formally equivalent to an Euler fluid under the Madelung transformation [5], it has a singular equation of state whose pressure includes the term $\nabla^2\sqrt{\rho}/\sqrt{\rho}$, where $\rho = |\psi|^2$. In the core region where ρ vanishes the pressure is singular, and the differentiation and manipulations required to derive Kelvin's theorem are not allowed. Thus there is no contradiction for any C. However, Kelvin's theorem does preclude reconnection in the 0° case, because one can consider a contour circling both vortices which remains outside the core, with an initial circulation $2h/m$, which cannot change.

To summarize, we have performed the first detailed simulations of a model for the core-scale interaction of quantized vortex filaments in helium II. Our results lend strong support to the hypothesis used in earlier studies of ensembles of vortices: If two vortices are anywhere near antiparallel when the large-scale fluid motion brings them together, they reconnect. Extensions of our methods to study the behavior of the core during pinning events, and perhaps the core dynamics of vortex rings, seem to be feasible calculations for the future.

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[1] R. P. Feynman, *Progress in Low Temperature Physics*, edited by C. Gortes (North-Holland, Amsterdam, 1955), Vol. 1.
[2] For a general introduction, see R. J. Donnelly, *Experimental Superfluidity* (University of Chicago Press, Chicago, 1967).
[3] K. W. Schwarz, Phys. Rev. B **31**, 5782 (1985); **38**, 2398 (1988).
[4] For a discussion of the local induction approximation in classical hydrodynamics, see S. E. Widnall, Ann. Rev. Fluid Mech. **1**, 141 (1975).
[5] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge Univ. Press, Cambridge, 1991).
[6] E. P. Gross, J. Math. Phys. **4**, 195 (1963).
[7] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].
[8] The actual values of $f(r)$ can easily be calculated by numerical integration.
[9] W. T. Ashhurst and D. I. Meiron, Phys. Rev. Lett. **58**, 1632 (1987).
[10] P. G. Saffman, J. Fluid Mech. **212**, 395 (1990).
[11] O. N. Boratav, R. B. Pelz, and N. J. Zabusky, Phys. Fluids A **4**, 581 (1992), and references therein.
[12] H. Nishimori and T. Nukii, J. Phys. Soc. Jpn. **58**, 563 (1988).
[13] E. P. S. Shellard, Nucl. Phys. **B282**, 624 (1987); see also R. Matzner, Comput. Phys. **2**, 51 (1988), for a gauge field version of this problem.
[14] I. Chuang, R. Durrer, N. Turok, and B. Yurke, Science **251**, 1336 (1991).
[15] T. Taha and M. Ablowitz, J. Comput. Phys. **55**, 203 (1984).