Interferometric Detection of Optical Phase Shifts at the Heisenberg Limit

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We show that the uncertainty in the relative quantum phase of two fields propagating in the arms of a Mach-Zehnder interferometer can be reduced to the Heisenberg limit by driving the interferometer with two Fock states containing equal numbers of photons. This leads to a minimum detectable phase shift far below that of any interferometer driven by a coherent light source.

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The interferometer is a fundamental apparatus in optical physics whose output signal is sensitive to the relative phase shift between two fields traveling down separated paths. The use of interferometers in optical gyroscopes [1] as well as gravitational wave detectors [2] relies on the ability to resolve extremely small relative shifts in the two path lengths with the smallest detectable shift in principle determined by the quantum properties of the illuminating field. Zero-point fluctuations in the laser and vacuum ports of the input beam splitter produce phase difference uncertainty between the fields propagating down the two paths. These phase fluctuations are indistinguishable from genuine changes in the path length difference of the two arms.

In the case of the two port interferometer with a coherent laser field and a vacuum field as inputs, the effect of zero-point fluctuations in the vacuum on the relative length measurement is amplified by the mean intensity of the laser [3–5]. During a measurement interval in which the laser supplies an average of n photons, the phase difference uncertainty between the two fields in the interferometer arms is $1/\sqrt{n}$ rad. Increasing the strength of the incident laser source does therefore increase the resolution of the device. Huge and expensive laser sources will, however, be required in order to resolve the small perturbations expected by the passage of a gravitational wave.

A possible mechanism for improving the sensitivity is to drive the interferometer with nonclassical states of light as the $1/\sqrt{n}$ level of relative phase fluctuations for a coherent source is well above the Heisenberg limit of $1/n$ rad [6]. To approach the Heisenberg limit it is necessary to introduce nonlocal quantum correlations between the photons in the two arms. One proposed scheme illuminates one of the input ports by a squeezed vacuum to reduce the vacuum fluctuations in the appropriate quadrature [3]. An alternative method is to coherently drive the optical fields with two strongly correlated atomic transitions [7].

In this paper, we show that the Heisenberg limit of sensitivity can be realized by driving the interferometer by two fields with no amplitude difference noise. Since the amplitude difference and phase difference between optical fields are Heisenberg conjugate variables, the quantum phase difference between these input fields is uncertain. We shall show, however, that the relative phase uncertainty between the output fields from the first beam splitter in the interferometer depends strongly only on the amplitude difference noise of the input fields.

Consider two classical fields with amplitudes $\alpha_{\rm in}$ and $\beta_{\rm in}$ and with absolute phases $\theta_{\rm in}$ and $\phi_{\rm in}$ incident on a 50/50 beam splitter as illustrated in Fig. 1. We assume the beam splitter introduces a phase shift of $\pi/2$ on reflection. The classical phase difference and intensity difference between the output fields are given by

$$\tan(\phi_{\rm out}-\theta_{\rm out}) = \frac{\alpha_{\rm in}-\beta_{\rm in}}{2\alpha_{\rm in}\beta_{\rm in}\cos(\phi_{\rm in}-\theta_{\rm in})}\,,$$
$$(\alpha_{\rm out})^2-(\beta_{\rm out})^2 = 2\alpha_{\rm in}\beta_{\rm in}\sin(\phi_{\rm in}-\theta_{\rm in})\,. \tag{1}$$

If the input fields have equal amplitude $\alpha_{\rm in}=\beta_{\rm in}$, the beam splitter produces output fields with zero phase difference, independent of the absolute and relative phase noise in the input modes. Alternatively, if the input fields have equal phase $\phi_{\rm in}=\theta_{\rm in}$, the two output ports generate light of equal intensity.

In analogy to the classical scheme analyzed above, we consider two Fock states with the same photon number m as inputs to a 50/50 beam splitter. We want to examine the nature of the quantum phase distribution of the out-

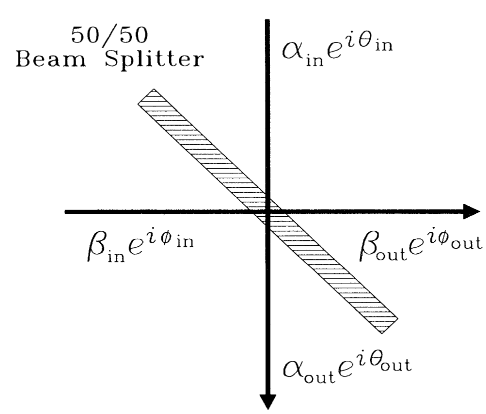

FIG. 1. Classical amplitudes and phases for the input and output fields of a beam splitter.

put fields from the beam splitter. To do this, we use the following basis of $s+1$ states $\{ |\theta_l\rangle \mid l = 0, \ldots, s \}$ of well defined phase [8,9] to describe the distribution for each mode,

$$|\theta_l\rangle = \frac{1}{\sqrt{s+1}} \sum_{p=0}^{s} e^{ipl\epsilon} |p\rangle , \tag{2}$$

where $\{ |p\rangle \mid p = 0, \ldots, s \}$ denotes the Fock states. The rotation between adjacent phase states is $\epsilon = 2\pi/(s+1)$. The corresponding basis for the two mode input field is then formed from the outer product $|\theta_l\, \theta_{l'}\rangle = |\theta_l\rangle \otimes |\theta_{l'}\rangle$. We denote the annihilation operators for the two modes by a and b. The probability that the phase difference between the output fields is $\Delta\theta$, where $\Delta\theta$ is an integral multiple of ϵ, can be found by applying the unitary transformation for the beam splitter and overlapping the result with the phase state basis. Finally, tracing over the possible values for the absolute phase we find

$$\begin{aligned} P(\Delta\theta) &= \sum_{l=0}^{s} \left| \langle \theta_l\, \theta_{l-(\Delta\theta)/\epsilon} | e^{i\frac{\pi}{4}(ab^\dagger + a^\dagger b)} | m\, m \rangle \right|^2 \\ &= \frac{1}{2^{2m}(s+1)} \left| \sum_{r=0}^{m} \sqrt{\frac{2(m-r)!}{m-r!^2}} \sqrt{\frac{2r!}{r!^2}}\, e^{2ir\Delta\theta} \right|^2 . \end{aligned} \tag{3}$$

Figure 2 illustrates this quantum phase distribution for $m = 50$ photons. The distribution is well localized around a phase difference of zero with a width at the Heisenberg limit of $1/(2m)$ rad. Since no phase origin is defined for either of the inputs, the absolute phase of the output fields must be completely uncertain. Localization of the relative phase variable indicates that the beam splitter correlates the phases of the photons at its output ports. For comparison we have overlaid the distribution generated by two coherent states with an average number of 50 photons but Poissonian photon statistics. The width of the resulting phase distribution is much wider than for the Fock states.

We now want to show how we can exploit this narrow phase difference distribution produced by Fock states in an interferometer. To do this we consider our beam splitter to be the first in a Mach-Zehnder arrangement as illustrated in Fig. 3. Providing an equal number of photons is injected into each of the two input ports, the relative phase uncertainty between the two fields in the arms of the interferometer will be at the Heisenberg limit. We consider input fields from a system which simultaneously produces one photon in each mode. In general this is described by the density operator

$$\rho = \sum_{nn'} c_{nn'} |n\, n\rangle\langle n'\, n'| . \tag{4}$$

Examples of sources of such states are two photon emission and nondegenerate parametric amplification which generates the two photon squeezed state.

The coherent mixing of the input fields by the interferometer is a unitary transformation which depends on the path length difference between the two arms. If we denote the path length difference by z, and let the wave number of the field be k, then the output state which is measured by the photodetectors can be calculated by applying the operator $\exp[kz(a^\dagger b - b^\dagger a)/2]$ to the input state. For the mixed state specified in Eq. (4), the photons divide equally between the two input ports so that counting a combined total of $2r$ photons at the two output ports specifies the input field as the dual Fock state $|r\, r\rangle$. Note that this requires measuring the sum current as well as the difference current from the pho-

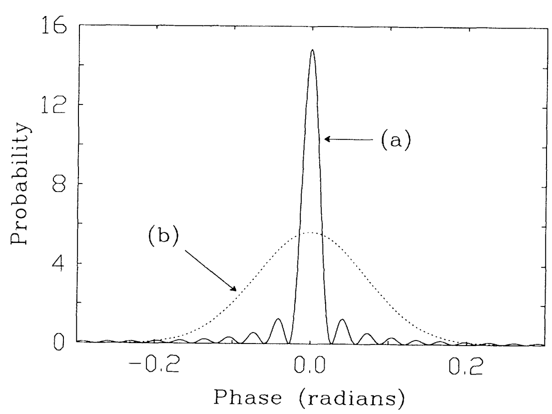

FIG. 2. (a) Overlap of the output field of a beam splitter illuminated by two number states containing 50 photons with the phase difference states. (b) The much larger phase uncertainty resulting from mixing two coherent states of mean 50 photons at the same beam splitter.

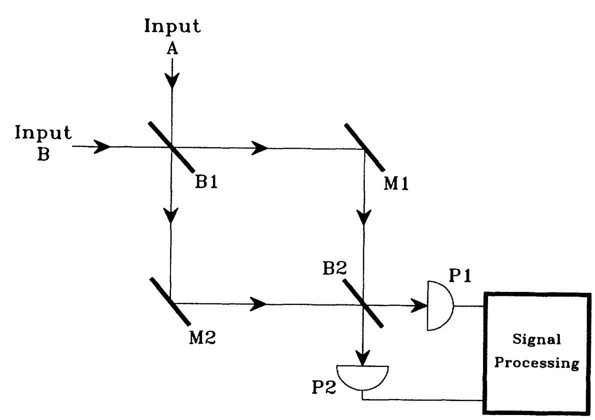

FIG. 3. The layout of a Mach-Zehnder interferometer. The sum current from the photon counting detectors P1 and P2 contains information about the total number of photons arriving in the input ports A and B. The difference current contains information about the relative length of the two possible paths from the 50/50 beam splitter B1 to the 50/50 beam splitter B2. M1 and M2 are reflecting mirrors.

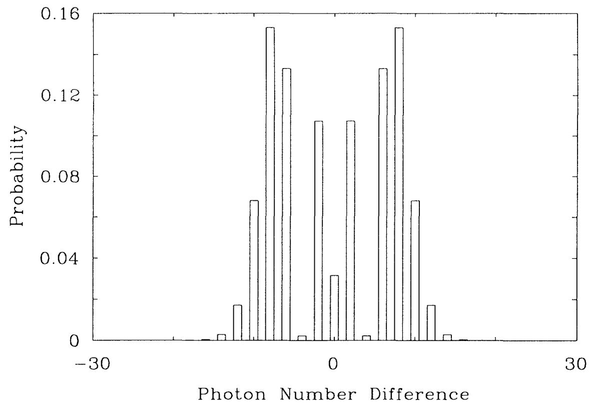

FIG. 4. The probability distribution for the photon number difference between the two output fields of a Mach-Zehnder interferometer. The input field was a dual Fock state with each input port receiving 5000 photons. The relative phase shift between the arms was 10^{-3} rad.

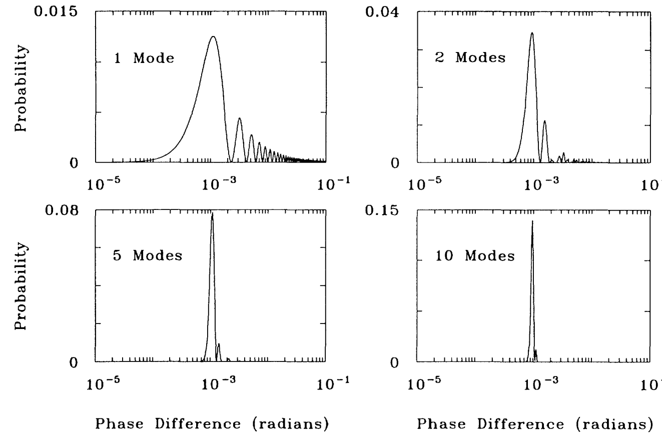

FIG. 5. Simulated probability distributions for the path length difference for a Mach-Zehnder interferometer containing 1, 2, 5, and 10 modes. The peaks of the distributions are around the actual path length difference of 10^{-3} rad and become narrower as information from more modes is combined.

todetectors. An alternative method with similar results is to use a pulsed scheme with very nearly the same photon number from pulse to pulse [10]. The minimum detectable phase shift is determined primarily by the mean illuminating field and is not sensitive to small fluctuations in the input intensity.

The probability distribution for the difference in the number of photons $2q$ measured at the two output ports for a phase difference kz is given by

$$P(2q|kz) = \left|\left\langle r-q\, r+q\right| e^{\frac{kz}{2}(a^\dagger b - b^\dagger a)} \left| r\, r\right\rangle\right|^2 . \quad (5)$$

If the path length difference is zero the interferometer transmits the input states without modification which leads to a zero photon number difference for the dual Fock state input. An imbalance in the photon number difference indicates a nonzero path length difference for the two arms. For $q \ll r$ the probability distribution for the difference count reduces to

$$P(2q|kz) = J_q^2(kzr) , \quad (6)$$

where J denotes the Bessel function. This is illustrated in Fig. 4 for $r = 5000$ photons and a phase difference of $kz = 10^{-3}$ rad. The width of this distribution is of order $|q| < kzr$. This reflects the fact that the relative phase noise in the arms is only limited by the Heisenberg uncertainty principle. We are therefore able to detect phase shifts close to the Heisenberg limit. To see how this might be done in a practical device, we shall now consider the dependence of the photon number difference distribution on the relative path length between the two arms. We shall see that we can derive information about the phase shift from the difference current between the photodetectors. The probability distribution for the relative path length after measurement of a particular photon number difference, $P(kz|2q)$, is given by

$$P(kz|2q) = \frac{1}{P(2q)} P(2q|kz)P(kz) . \quad (7)$$

The fraction $1/P(2q)$ is a normalization factor and $P(kz)$ denotes the prior knowledge about kz. Note that information from several spatially independent modes in the interferometer can be combined. In Fig. 5, we illustrate possible probability distributions for kz after combining the results of 1, 2, 5, and 10 modes. The procedure adopted is as follows. We consider a Mach-Zehnder interferometer with a phase difference corresponding to 10^{-3} rad. Each input mode is taken to be in a Fock state containing 5000 photons. Using the probability distribution in Eq. (6) and a random number we then simulate a particular photon number difference between the detectors at the output ports. Applying Eqs. (6) and (7) we generate a probability distribution for kz. For simplicity we have assumed a flat prior for the first mode. Information from subsequent modes is combined by updating the prior.

In Fig. 6(a), we simulate the measurement of a phase shift of 10^{-3} rad by a five mode interferometer for various amplitudes of the input dual Fock states. For each choice of the input field strength, we select a particular photon number difference for each mode. Using these simulated measurements we construct the probability distribution for the relative path length and calculate the mean value and standard error. Providing the total photon number is greater than the Heisenberg limit of 1000 photons, the phase shift can be distinguished from the zero position of the interferometer. In Fig. 6(b), we show a similar simulation for a classical interferometer. Injected into the input ports are a vacuum field and a coherent laser

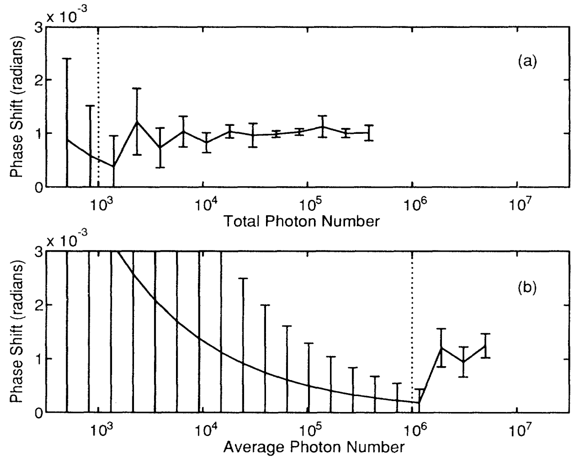

FIG. 6. (a) Simulations of the measurement of a phase shift of 10^{-3} rad for a five mode interferometer driven by dual Fock states of various amplitudes. We plot the mean value of the probability distribution for the phase shift derived from the simulated measurements and use error bars to illustrate the standard error. Note the phase shift is distinguishable from zero for input photon numbers down to the Heisenberg limit (dotted line). (b) As for (a) but with the interferometer driven by a coherent field (see text). The phase shift is resolved only if the input intensity is above the classical limit (dotted line).

field. The zero position of the interferometer is adjusted so that no photons exit from one of the output ports for zero phase shift. The phase shift of 10^{-3} rad is then detected by the presence of photons in this port. The result of the simulation shows that the classical driving field must be significantly stronger in order to detect the relative path length change. For a coherent field supplying less than 10^6 photons during the detection period, most measurements record no photons in the dark port and the phase shift cannot be distinguished from zero.

We have presented a new scheme for measuring phase shifts at the Heisenberg limit. The important property of the input field is found to be equal numbers of photons in the two ports. If this is satisfied nonlocal quantum correlations are generated between the fields in the two arms of the interferometer. This allows greater resolution than for any interferometer driven by a coherent field. Although the two mode squeezed state satisfies this required input field criteria, the most energy efficient state is the dual Fock state.

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