# Coherent Population Trapping States of a System Interacting with Quantized Fields and the Production of the Photon Statistics Matched Fields

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I calculate the coherent population trapping states of a three-level system interacting with two quantized fields. I show that such trapping states have novel properties. For example, I show that the trapping states for a Λ system involve the production of radiation fields with matched photon statistics. I discuss a method for producing these trapping states. For a ladder system, the trapping states even lead to the nonclassical character of the radiation fields.

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The coherent population trapping states of a Λ system interacting with two coherent fields applied on different transitions are well known [1–6]. These are the stationary states of the Hamiltonian which remain nonevolving in the presence of the radiative relaxation of the system. Thus a collisionless Λ system irradiated by two intense fields can lead to a nonevolving state under certain conditions on the detunings and relaxation parameters. Such trapping states have been extensively studied and have been utilized in a number of different contexts such as lasing without inversion [7,8] and the induced transparency of the medium [9,10]. In the previous studies the driving fields have been treated *classically* and thus the external fields [11] are taken to be *prescribed* and *undepleted* in the process of interaction. It would be important to understand the dynamics of the coupled atom-field system by *including the evolution* of the fields. Thus one has to investigate the possibility of the trapping states of quantized fields interacting with atoms. This would also be necessary in the context of nonclassical fields interacting with atoms, which leads to some unusual properties [12].

In this Letter I derive the coherent population (to be abbreviated as *CPT* states) trapping states of a Λ system interacting with two quantized fields. I demonstrate the connection and relevance of such trapping states to recent ideas on the amplification without inversion [13], transparency, and matched pulses [10]. I further demonstrate how one can produce radiation fields with not only matched amplitudes but with *matched photon statistics.* Finally I present some results for other types of three-level systems. The trapping state in, say, a ladder system can even exhibit nonclassical characteristics.

Consider the interaction of a Λ system with a two-mode quantized field. Let the mode with frequency $\omega_a$ ($\omega_b$) characterized by the annihilation and creation operators $a$ and $a^\dagger$ ($b$ and $b^\dagger$) interact with the transition $|1\rangle\leftrightarrow|3\rangle$ ($|1\rangle\leftrightarrow|2\rangle$). In the interaction picture the total Hamiltonian can be written in the form

$$H=\hbar(g_a a|1\rangle\langle 3|+g_b b|1\rangle\langle 2|+\text{H.c.})+\hbar\Delta_1|1\rangle\langle 1|+\hbar(\Delta_1-\Delta_2)|2\rangle\langle 2|\,, \tag{1}$$

where $g$'s are the coupling constants and Δ's are the detunings,

$$\Delta_1=\frac{E_1}{\hbar}-\omega_a,\quad \Delta_2=\frac{E_1-E_2}{\hbar}-\omega_b,\quad E_3\equiv 0\,. \tag{2}$$

If the fields $a$ and $b$ are treated as *classical prescribed* numbers $\alpha$ and $\beta$, then in the special case $\Delta_1=\Delta_2$ the state

$$|\psi_0\rangle=\mathcal{N}(g_b\beta|3\rangle-g_a\alpha|2\rangle) \tag{3}$$

is an eigenstate of $H_c$:

$$H_c|\psi_0\rangle=0;\quad H\to H_c,\quad a\to\alpha,\quad b\to\beta\,. \tag{4}$$

In (3) $\mathcal{N}$ is fixed by normalization. The state (3) is stable against spontaneous emission since both the states $|2\rangle$ and $|3\rangle$ are stable states. Thus the state $|\psi_0\rangle$ does not evolve in time. The situation is different if one includes effects of collisions [14] and finite temperature.

I now derive the *CPT* states of the quantized Hamiltonian (1). I impose the condition $\Delta_1=\Delta_2$ and try a solution of the form

$$|\psi\rangle=\mathcal{N}(g_b\beta|3\rangle-g_a\alpha|2\rangle)|\psi_f\rangle\,, \tag{5}$$

where $\alpha$ and $\beta$ are *unknown* coefficients and where $|\psi_f\rangle$ is the wave function corresponding to the modes $a,b$ of the field. On substituting (5) in the eigenvalue equation $H|\psi\rangle=0$, I obtain the condition

$$(a\beta-b\alpha)|\psi_f\rangle=0\,. \tag{6}$$

The general solution of (6) can be obtained in terms of the coherent states [15] $|z_a,z_b\rangle$ associated with the two modes of the field. Thus I find that the most general solution to the eigenvalue problem $H|\psi\rangle=0$ is

$$|\psi\rangle=\mathcal{N}(g_b\beta|3\rangle-g_a\alpha|2\rangle)\int q(z)\left|z,\frac{z\beta}{\alpha}\right\rangle d^2z\,. \tag{7}$$

Here $\alpha$, $\beta$, and $q(z)$ are unknown parameters to be fixed by the time evolution of the system. The states (7) are the *CPT* states of the quantized system. These states are *stable against spontaneous emission* just like the state (3). The classical solution (3) is obtained from the quan-

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tized result (7) by ignoring the dynamic evolution of the fields. This is achieved by choosing $q(z)=\delta^{(2)}(z-\alpha)$.

I next examine some general characteristics of the *CPT* states of the quantized system. From (7) the reduced density matrix for the field modes is found to be

$$\rho^{(f)}=\int\int d^2z_1 d^2z_2 q(z_1)q^*(z_2)\left|z_1,\frac{z_1\beta}{\alpha}\right\rangle\left\langle z_2,\frac{z_2\beta}{\alpha}\right| \tag{8}$$

from which density matrices for modes $a$ and $b$ can be obtained. In the special case when $\beta/\alpha=\beta^*/\alpha^*=1$, one gets

$$\rho^{(a)}=\int\int d^2z_1 d^2z_2 q(z_1)q^*(z_2)|z_1\rangle\langle z_2|\exp(z_1z_2^*-\tfrac{1}{2}|z_1|^2-\tfrac{1}{2}|z_2|^2)=\rho^{(b)}. \tag{9}$$

Note that (9), in general, represents a mixed state and that the *photon statistics of the two modes becomes matched.* Note further that in general

$$\rho^{(f)}\neq\rho^{(a)}\rho^{(b)}. \tag{10}$$

The possible exception is when $q(z)=\delta^{(2)}(z-z_0)$. Thus the two modes become *correlated.*

These general considerations give us a whole class of the *CPT* states of the quantized system. The values of the unknown $\alpha$ and $\beta$ and the function $q(z)$ can only be fixed by the initial conditions and by the physical situation at hand. In order to illustrate the utility of the ideas on the trapping states of quantized systems, I consider explicitly some cases and also establish connection with some recent works.

I consider a complete quantum treatment of the model system shown in Fig. 1(a). A semiclassical description of this model has been given by Harris [10]. I will assume that the fields on the two transitions have large coherent components which can be treated semiclassically. I will further assume that these components are resonant with the respective transitions. The quantized modes may have *different frequencies.* In the interaction picture the Hamiltonian for this system can be written in the form

$$H=-g_a(\alpha+ae^{-i\delta_a t})A_a^\dagger-g_b(\beta+be^{-i\delta_b t})A_b^\dagger+\text{H.c.},$$
$$A_a^\dagger=|1\rangle\langle 3|,\quad A_b^\dagger=|1\rangle\langle 2|, \tag{11}$$
$$\delta_a=\omega_a-\omega_1,\quad \delta_b=\omega_b-\omega_2,$$

where $g_a$ and $g_b$ are the coupling constants. Harris treated the dynamic evolution of modes $a$ and $b$ classically and proved a remarkable result: The system leads to the generation of nonzero steady state fields such that $a\beta=b\alpha$ even though the sample length may exceed several absorption lengths. This happens for all modes such that $\delta_a=\delta_b$. We now derive the result of a full quantum calculation. The density matrix equations for the combined atom-field system can be obtained using (11) and the decay of the excited state $|1\rangle$ to the levels $|2\rangle$ and $|3\rangle$. From this equation I derive a master equation [16] for the reduced density matrix of the field modes $a$ and $b$. In the calculations (the details to be presented elsewhere) I treat the *coherent components to all orders* but I derive the dynamical equations to second order in $g_a$ and $g_b$. I also assume rapid decay of the atomic excited state compared to the time scale over which the field evolves. The final result for the evolution of the field density matrix is

$$\dot\rho=-(a^\dagger a\rho-a\rho a^\dagger)|g_a|^2g_b\beta C_a(-i\delta_a)$$
$$-(a^\dagger b\rho-b\rho a^\dagger)|g_a|^2g_b(-\alpha)C_a(-i\delta_b)e^{i(\delta_a-\delta_b)t}+\text{H.c.}+\text{ terms with } a\leftrightarrow b,\alpha\leftrightarrow\beta. \tag{12}$$

The coefficient $C_a$ is related to the linear susceptibility [17] of the atomic system driven by the fields $\alpha$ and $\beta$ and is a function of the detuning $\delta_a$. Thus $C_a$ will depend on all orders of $\alpha$ and $\beta$. This is clearly seen from the equation for the mean value of $a$:

$$\langle\dot a\rangle=-\langle a\rangle|g_a|^2g_b\beta C_a(-i\delta_a)+\langle b\rangle|g_a|^2g_b\alpha C_a(-i\delta_b)e^{i(\delta_a-\delta_b)t}. \tag{13}$$

If the difference $\delta_a-\delta_b$ is large, then the coupling between $a$ and $b$ modes can be ignored. However, for $\delta_a=\delta_b$ very interesting results emerge. First of all at the level of mean fields (13) predicts that

$$\langle a\rangle\beta=\langle b\rangle\alpha. \tag{14}$$

Thus apart from a scaling factor one produces fields with matched amplitudes, a case discussed by Harris in his semiclassical analysis. Note that in the absence of the coherence mediated coupling between $a$ and $b$, the steady state amplitude becomes zero because no population inversion is imposed in field propagation. Thus the coherence of the trapping state leads to nonzero and matched

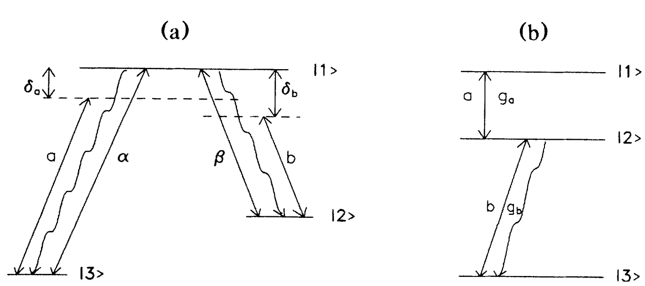


FIG. 1. Schematic illustration of the energy levels for (a) the Λ system and (b) the ladder system, and the different interactions with the quantized fields.

fields. The density matrix equation (12) can also be solved analytically and then one can investigate the photon statistics of the generated fields.

Using the conservation laws and the techniques based on the Glauber-Sudarshan $P$ function [15,18], I have found the following steady state solution of (12):

$$\rho=\int P(z_1,z_2)|z_1,z_2\rangle\langle z_1,z_2|d^2z_1d^2z_2\,,$$
$$P(z_1,z_2)=\delta^{(2)}\left(z_2-\frac{z_1\beta}{\alpha}\right)\int P_0(\mathscr{z}_1,\mathscr{z}_2)d^2\mathscr{z}_1d^2\mathscr{z}_2\delta^{(2)}\left[z_1-\left(\mathscr{z}_1+\frac{g_a^*C_a\mathscr{z}_2}{g_b^*C_b}\right)\left(1+\frac{\beta}{\alpha}\frac{g_a^*C_a}{g_b^*C_b}\right)^{-1}\right], \tag{15}$$

where $P_0$ is the $P$ function associated with the input fields. From (15) we find the following: (a) The solution is in the form (8) obtained from general considerations on *CPT* states of quantized systems. (b) The degeneracy of the solution has been fixed by the initial conditions. (c) The system leads to the production of the *quantum statistics matched fields.* This last result follows from the overall delta function outside the integral sign in (15). (d) Note further that if $\delta_a\neq\delta_b$, then the steady state solution of (12) will be a vacuum state $\rho=|0,0\rangle\langle 0,0|$ as one would expect from the quantum version of Beer's law. I have also obtained the *time-dependent solution of (12)* which I do not give here. It may be noticed that one can produce fields with matched statistics for all pairs of modes such that $\delta_a=\delta_b$. The frequency of the individual mode is irrelevant. I mention that a master equation of the form (12) has also been derived previously in the context of a model of amplification. The above results for the generation of fields with matched statistics will also be applicable to this amplifier model [13].

I finally consider the possibility of trapping like states in ladder systems. The Hamiltonian for the scheme of the Fig. 1(b) is

$$H=(g_a|1\rangle\langle 2|a+g_b|2\rangle\langle 3|b+\text{H.c.})\,. \tag{16}$$

For the sake of simplicity I assume that the fields are resonant with the respective transitions. If $a$ and $b$ are treated classically with amplitude $\alpha$ and $\beta$, then the state

$$|\psi\rangle=(\beta g_b|1\rangle-\alpha^*g_a^*|3\rangle) \tag{17}$$

is an eigenstate of $H$ with zero eigenvalue. However, such a state is *not* stable against radiative decay unless both states $|1\rangle$ and $|3\rangle$ do not decay. For the quantized Hamiltonian (16) the states with $H|\psi\rangle=0$ can be obtained by choosing

$$|\psi\rangle=(\beta g_b|1\rangle-\alpha^*g_a^*|3\rangle)|\psi_f\rangle\,, \tag{18}$$

where $|\psi_f\rangle$ is the solution of the eigenvalue equation

$$(\beta a^\dagger-\alpha^*b)|\psi_f\rangle=0\,. \tag{19}$$

One *possible* solution to the eigenvalue problem is the squeezed vacuum state [19]

$$|\psi_f\rangle=\sum_n\left(\frac{\beta}{\alpha^*}\right)^n|n,n\rangle\,. \tag{20}$$

Thus the trapping state associated with a ladder system interacting with a quantized field is the direct product of the state (17) and the squeezed vacuum state. The state (17) or (18) is meaningful provided that the relaxation mechanism is such that it is stable against decay. A more general solution of (19) can be written in terms of the two-mode squeezing operator [19] $S$ and an arbitrary state for mode $a$:

$$|\psi_f\rangle=S|\phi_a,0\rangle\,. \tag{21}$$

The $\phi_a$ content of the state will be fixed by the initial conditions. For a ladder system, the reduced density matrix equation for the fields can be obtained by following analysis similar to that leading to (12). This analysis shows the kind of states [like (21)] that can be generated.

The foregoing analysis [20] shows the remarkable properties of the *CPT* states of atoms interacting with quantized fields. I have also shown how one can produce such trapping states. I have shown how the idea of the transparency and matched pulses follows from the existence of such trapping states. In the foregoing I have only treated the three-level systems but it is clear how one can study the coherent population trapping states of a multilevel system interacting with quantized fields.

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