Topological Closed-String Interpretation of Chern-Simons Theory

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The exact free energy of SU(N) Chern-Simons theory at level k is expanded in powers of $(N+k)^{-2}$. This expansion keeps rank-level duality manifest, and simplifies as k becomes large, keeping N fixed (or vice versa)—this is the weak-coupling (strong-coupling) limit. With the standard normalization, the free energy of the three-sphere in this limit is shown to be the generating function of the Euler characteristics of the moduli spaces of surfaces of genus g, providing a string interpretation for the perturbative expansion. A similar expansion is found for the three-torus, with differences that shed light on contributions from different spacetime topologies in string theory.

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The perturbative expansion of any quantum field theory (QFT) with fields transforming in the adjoint representation of SU(N) is a topological expansion [1] in surfaces, with N^{-2} playing the role of a handle-counting parameter [2]. For N large, one hopes that the dynamics of the QFT is approximated by the sum (albeit largely intractable) of all planar diagrams. The topological classification of diagrams has nothing *a priori* to do with approximating the dynamics with a theory of strings evolving in spacetime.

Gross [3] (see also Refs. [4,5]) has shown recently that the large N expansion does actually provide a way of associating a theory of strings in two-dimensional QCD. Maps of two-dimensional string world sheets into two-dimensional spacetimes are necessarily somewhat constricted. What one would like is a QFT with fields transforming in the adjoint representation in $d>2$, which is at the same time exactly solvable. One could then, in principle, attempt to associate a theory of strings with such a QFT by exhibiting a "sum over connected surfaces" interpretation for the free energy of the QFT. There is no guarantee that such an association will exist.

Chern-Simons theory in three dimensions is precisely such a *rara avis* among QFTS. It is described by a functional integral

$$Z[M]\equiv\int DA\exp(ikI_{\rm CS})\,,$$

with

$$I_{\rm CS}\equiv(1/4\pi)\int_M d^3x\,{\rm tr}[A\,dA+\tfrac{2}{3}A^3]\,.$$

Here M is a closed oriented 3-fold, and the gauge group is assumed simply connected so the principal G bundle on which A is a connection can be trivialized. The trace is normalized so k is an integer. For $G=$SU(N), the case considered in detail in this Letter, the trace is taken in the defining representation. In the standard large N limit, $k\propto N$. The functional integral will not be used in the following. Instead, I shall exploit Witten's [6] relation of Chern-Simons theory to knot theory and conformal field theory.

I shall show in this Letter that the exact free energy of SU(N) Chern-Simons theory at level k has an expansion that admits a string interpretation. This expansion is obtained in two steps, the first an expansion in $(k+N)^{-2}$, and the second a "double-scaling" limit [7]. The expansion in $(k+N)$ is natural from the point of view of rank-level duality [8]. When M is the three-sphere, S^3, the scaled free energy turns out to be the generating function of the Euler characteristics of the moduli spaces of surfaces with g handles [9]. I shall also compute the expansion when M is the three-torus, T^3. This can be interpreted as a sum over surfaces with one boundary (or puncture). An important comment: The normalization of the partition functions used in this Letter is that used in Chern-Simons theory, and may be inappropriate for string identifications. For example, comparison with Casson's invariant suggests that the free energy on S^3 should be chosen to vanish.

To start, Witten [6] showed that

$$Z[S^3]=S_{0,0}\,,$$

with the normalization $Z[S^2\times S^1]=1$. Here S_α^β is the modular transformation matrix representing the action of the modular group SL(2,$\mathbb{Z}$) on the characters of the SU(N) Kac-Moody algebra at level k:

$$\chi_\alpha(-1/\tau)=\sum_\beta S_\alpha^\beta\chi_\beta(\tau)\,.$$

The characters of the Wess-Zumino-Witten model are known, so S can be determined without resorting to computations with the functional integral definition of the Chern-Simons theory. Thus for SU(N) [10], we have

$$Z[S^3,N,k]=(k+N)^{-N/2}\left(\frac{k+N}{N}\right)^{1/2}\prod_{j=1}^{N-1}\left\{2\sin\left(\frac{j\pi}{N+k}\right)\right\}^{N-j}.$$

The expansion and scaling limit undertaken below lead to a result that can be deduced simply by taking a large k limit

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of $Z[S^3,N,k]$. The longer route makes contact with the idea of double scaling [7] and keeps rank-level duality manifest in the scaling limit. Define $M \equiv k+N$, and $x \equiv N/M$. The rank-level duality [8] under an interchange of N and k, or equivalently $x \leftrightarrow 1-x$, is easily checked:

$$\frac{Z[S^3,N,k]}{Z[S^3,k,N]}=\left(\frac{1-x}{x}\right)^{1/2}.$$

The normalization of the partition functions could be changed to give exact duality; however, the normalization we use is natural from the point of view of Kac-Moody algebras. The planar term in the large N expansion with x fixed is

$$\ln Z \sim N^2 \int_0^1 dy(1-y)\ln 2\sin(\pi y x)\,,$$

obtained by Camperi, Levstein, and Zemba [11].

Recall $\sin(\pi z)=\pi z\prod_{n=1}^{\infty}(1-z^2/n^2)$. It follows that

$$Z[S^3,N,k]=M^{-[(Mx)^2-1]/2}(2\pi)^{Mx(Mx-1)/2}G(Mx+1)\exp(-F_0)\,,$$

where G is Barnes' G function [12], and

$$F_0 \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{Mx-1}\sum_{m=1}^{\infty}\frac{1}{m}(Mx-j)\left(\frac{j}{nM}\right)^{2m}=\sum_{m=1}^{\infty}\frac{1}{m}\frac{\zeta(2m)}{M^{2m}}\sum_{j=1}^{Mx-1}(Mx-j)j^{2m}.$$

The sum over j can be written in terms of the Bernoulli polynomials [12],

$$\phi_k(z)\equiv z^k-\frac{k}{2}z^{k-1}+C_2^kB_1z^{k-2}-C_4^kB_2z^{k-4}+\cdots \quad \text{up to } z \text{ or } z^2,$$

which satisfy

$$\sum_{j=1}^{L-1} j^{m-1}=\frac{1}{m}\phi_m(L)\,.$$

After some rearrangement, we arrive at

$$F_0=M^2\sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\frac{x^{2m+2}}{(2m+1)(2m+2)}-\sum_{k=1}^{\infty}M^{2-2k}(-)^kB_k\frac{2k-1}{2k!}\frac{d^{2k-2}}{dx^{2k-2}}\ln\left(\frac{\sin\pi x}{\pi x}\right).$$

The asymptotic behavior of Barnes' G function is

$$\ln G(z+1)\sim\frac{z}{2}\ln 2\pi-\frac{3z^2}{4}+\frac{z^2}{2}\ln z-\frac{B_1}{2}\ln z+\sum_{r=2}^{\infty}(-)^{r-1}\frac{B_r}{2r(2r-2)}z^{2-2r}.$$

We can now put all the terms together, noting that the $\ln\pi x$ terms from F_0 cancel against terms coming from the G function. This renders the rank-level duality manifest in the higher genus contributions. The end result, if we fix $N \equiv Mx$, and take $M\uparrow\infty$, is

$$F[S^3,N]=\tfrac{1}{2}\ln x+N^2[\tfrac{3}{4}-\tfrac{1}{2}\ln 2\pi x]+\frac{B_1}{2}\ln N+\sum_{k=2}^{\infty}N^{2-2k}(-)^k\frac{B_k}{2k(2k-2)}.$$

The (virtual) Euler characteristic of surfaces with g handles [9] is

$$\chi_g=(-)^g\frac{B_g}{2g(2g-2)}.$$

Thus, even including the scaling violations evident in the $N^2\ln x$ term [13], the free energy we have computed is precisely the generating function of Euler characteristics. Thus we have found a string interpretation of the free energy of Chern-Simons theory, since the Euler class is a natural measure on moduli space, and one way of characterizing string theories is to give measures on moduli spaces for all genus.

I turn now to the three-torus T^3. Witten [6] showed that $Z[T^3,N,k]$ counts the number of integrable irreducible highest-weight representations of the affine Kac-Moody algebra at level k. These are in one-to-one correspondence with Young tableaux with at most $N-1$ rows and at most k columns. We thus find

$$Z[T^3,N,k]=\frac{1}{kB(k,N)}\,,$$

where B is the Euler beta function. Observe that $Z[T^3,N,k]/Z[T^3,k,N]=x/1-x$, similar to the analogous ratio for S^3. The B function can be written as

$$B(M(1-x),Mx)=x^{Mx-1/2}(1-x)^{M(1-x)-1/2}(2\pi/M)^{1/2}\exp C(M,x)\,,$$

with

$$C(M,x)=2\int_0^\infty \frac{dt}{e^{2\pi t}-1}\arctan\left[\frac{(tM^{-1})^3+tM^{-1}(1-x+x^2)}{x(1-x)}\right].$$

In the limit $Mx\equiv N$ fixed, $M\to\infty$, using $B_k=4k\int_0^\infty dt\, t^{2k-1}(e^{2\pi t}-1)^{-1}$, and $\arctan(z)\approx z-z^3/3+z^5/5-\cdots$ $(z\approx 0)$, we find

$$F[T^3,N,k]\sim -\frac{1}{2}\ln\left[\frac{N}{2\pi}\right]-N(1-\ln x)+\sum_{k=1}^{\infty}(-)^k N^{1-2k}\frac{B_k}{2k(2k-1)}\,.$$

Thus the expansion should be interpreted as a sum over surfaces with one puncture or hole. However, the Euler characteristic of surfaces with m punctures and g handles is

$$\chi_{g,m}=(-)^m\frac{\Gamma(2g-2+m)}{\Gamma(m+1)\Gamma(2g-2)}\chi_g\,;$$

thus the coefficient of N^{1-2k} is not as we would have expected for surfaces with a simple puncture.

Observe the $\ln N$ term in $F[T^3]$, the only term that does not come from a surface with just one puncture. Like the odd powers of N found in $F[T^3]$ and the $\ln N$ term in $F[S^3]$, it is not what we would expect from general arguments [1]. These general arguments, however, apply only to the perturbative evaluation of large N functional integrals. In the present case (assuming a normalization where $F[S^3]=0$, since S^3 has only a trivial flat connection), the functional integral must be decomposed into a finite-dimensional integral over flat connections (dm) and a functional integral over fluctuations about these flat connections:

$$\int DA\exp(ikI_{\rm CS}[A])=\int dm\exp(ikI_{\rm CS}[m])\int Da\exp(ikI_{\rm CS}^{\rm pert}[a;m])\,.$$

The moduli space of flat connections depends on N, and one must also evaluate the value of the action at these flat connections. These conceptually simple steps cannot at this time be carried out explicitly. However, it should be clear that anomalous N dependence should be expected on manifolds (such as T^3 and $S^2\times S^1$) with nontrivial moduli spaces of flat connections.

Why is the appearance of a boundary natural? Every 3-fold can be obtained from S^3 by cutting out a tubular neighborhood of links embedded in S^3, and then glueing the tubular neighborhood back after acting on its boundary by a diffeomorphism—this is called surgery. T^3 can be obtained from S^3 by surgery on Borromean rings embedded in S^3. We would ideally wish to associate terms in the perturbative expansion on T^3 with surfaces whose boundaries are the reglued Borromean rings in S^3. However, the Borromean rings are linked and there is no simple way (without self-intersection) that one can make surfaces with Borromean ring boundaries.

It is of great interest to understand the Chern-Simons string theory for different 3-folds since it may teach us how to obtain a formulation of string theory independent of spacetime topology. It should be possible to explicitly compute the free energy on many simple 3-folds following the general results found by Witten [6]. For example, if one defines a function

$$\Xi(s)\equiv\sum S_{0,a}^{-s}\,,$$

where the sum runs over all integrable irreducible highest weight representations, then $\Xi(2g-2)$ is the partition function on $\Sigma_g\times S^{-1}$. Jeffrey's [14] Poisson resummation method should be useful in this analysis. For some idea of the computations involved, see Ref. [15]. Witten gave an open string interpretation to Chern-Simons theory [16]. The appearance of the Euler characteristics in the present work seems unrelated to the explicit geometry of his construction, but this merits further investigation.

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