Casimir Forces between Beads on Strings

Eric D'Hoker*
Physics Department, University of California, Los Angeles, Los Angeles, California 90024-1547

Pierre Sikivie†
Physics Department, University of Florida, Gainesville, Florida 32611
(Received 25 January 1993)

We consider a string with uniform energy density and tension and with a number of pointlike masses attached at fixed interdistances. We evaluate the effective interaction forces between these masses induced by the quantum fluctuations of the string. For interdistances large compared to the thickness of the string and small compared to the total length, these forces are universal and attractive and fall off for large distances as $1/r^3$ and for small distances as $1/r$. The attractive nature of these forces creates an instability under which masses added to the string tend to aggregate.

PACS numbers: 11.17.+y

In most dynamical models of string evolution, it is assumed from the outset that the energy density and tension are constant along the string. It is within this framework that the Nambu-Goto [1] and Polyakov [2] models are formulated, which are the basis of the string theory of elementary particles and their interactions. Also the extension to include elastic interaction energy that opposes bending of the string [3] and the models considered in [4] make this assumption.

There may be applications of string dynamics where this assumption is too stringent. The energy density and tension generally vary along strings which have zero modes attached to them, or strings which are wiggly on scales much smaller than the scale of interest to the observer. In particular, such may be the case with QCD strings, cosmic gauge strings, and the vortices that occur in superconductors and superfluids. Also, in applications to polymer physics, the mass distribution along a string is generally uneven. A natural extension of the Nambu-Goto model was proposed at the classical level in which new degrees of freedom on the string allow for variable energy density and string tension [5].

In the present paper, we examine the forces induced by the transverse quantum mechanical fluctuations of the string on deviations away from constant energy density. Inhomogeneities in the energy density of the string alter the frequency spectrum of the transverse string oscillations. This results in Casimir type forces between the inhomogeneities. In this Letter, we evaluate these forces in the case of pointlike masses attached to a straight string.

Part of our motivation is a possible connection with gravity. We will find below that when two inertial masses, say m_1 and m_2, are attached to the string, quantum fluctuations are responsible for an attractive force between the masses which is proportional to the product $m_1 m_2$ in the limit of small masses. The force does not scale like $1/r^2$ as in gravity, but it is proportional to the product of inertial masses, so that an analog of the equivalence principle holds here, which arises naturally from quantum mechanical effects.

Our assumptions are as follows. We consider a straight string of length L and thickness $1/\Lambda$, with constant energy density ρ and constant tension κ. N masses m_i are attached to the string in various fixed locations with coordinates x_i $(i=1,\ldots,N)$. Distances between the various masses will always be taken to be much smaller than the total length L [6], but much larger than the thickness $1/\Lambda$. In this limit, the Casimir forces will be found to be independent of both L and Λ. Only the $d-1$ transverse degrees of oscillation of the nonrelativistic string in d space dimensions are retained. We include a tension term in the action, but we shall neglect effects due to the bending of the string. The dynamics are thus assumed to be governed by

$$S=\int dt\left\{\int_0^L dx\frac{1}{2}(\rho\dot{\varphi}^2-\kappa\varphi'^2)+\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\varphi}^2(x_i)\right\}, \tag{1}$$

where φ stands for the $d-1$ transverse oscillation degrees of freedom.

We will encounter some quantities which require regularization. This may be effected by a cutoff on the wave vectors of the transverse string oscillations. We may in some applications think of this cutoff as physical in nature. At momenta beyond the cutoff, one would be probing the internal structure of the string, which typically involves new physics, not summarized by Eq. (1). Thus the ultraviolet cutoff should be of the order of the inverse thickness of the string, and we take its value to be Λ.

First, we derive a general equation for the frequencies of the oscillation eigenmodes. Let us label masses m_i such that their locations on the string are ordered: $0=x_0<x_1<x_2<x_3<\cdots<x_N<x_{N+1}=L$. The equation for a mode with frequency ω is derived from (1):

$$\kappa\varphi_\omega''(x)+\rho\omega^2\varphi_\omega(x)=-\omega^2\sum_{i=1}^{N}m_i\varphi_\omega(x_i)\delta(x-x_i). \tag{2}$$

0031-9007/93/71(8)/1136(4)$06.00

Within each interval between the masses, φ is just a sum of two exponentials:

$$x_{j-1} \le x \le x_j, \quad \varphi_\omega(x) = A_j(k)e^{ikx} + B_j(k)e^{-ikx}, \tag{3}$$

where $\rho\omega^2 = \kappa k^2$. Continuity of φ_ω and the known discontinuity of φ'_ω lead to the following transfer equations ($i = 1, \ldots, N$):

$$\begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = T_i \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = T_0 \begin{pmatrix} A_{N+1} \\ B_{N+1} \end{pmatrix}. \tag{4}$$

Here we define the transmission matrices:

$$T_j \equiv \begin{pmatrix} 1+t_j & t_j e^{-2ikx_j} \\ -t_j e^{+2ikx_j} & 1-t_j \end{pmatrix}, \qquad T_0 \equiv \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix}, \tag{5}$$

with $t_j = ikm_j/2\rho$. The equation for the momenta k is the condition that the system of Eqs. (4) admit nonzero solutions:

$$\det(1 - T_N T_{N-1} \cdots T_1 T_0) = 0. \tag{6}$$

We want to solve Eq. (6) and sum the corresponding oscillation frequencies. We will encounter infinities (i.e., quantities which are infinite in the limit $\Lambda \to \infty$) but they can all be absorbed into renormalizations of the parameters that have been introduced so far: energy density ρ, tension κ, and the masses m_i of the extra particles on the string. After these renormalizations have been effected, we will evaluate the interaction forces between the masses on the string.

When no masses are present ($N=0$), the momenta are given by $k_n^\pm L = 2\pi n$ for $n = 0,1,2,\ldots$. There are two linearly independent modes with the same frequency for every positive n. When masses are added to the string, this degeneracy is lifted as will be described below. The zero point energy that results from summing the eigenfrequencies is proportional to L and amounts to a renormalization of the effective energy per unit length ρ:

$$\delta\rho = +2\pi\hbar \left(\frac{\kappa}{\rho}\right)^{1/2} \Lambda^2 (d-1). \tag{7}$$

When a single mass m is present ($N=1$), Eq. (6) becomes

$$0 = 2 - 2\cos kL + \frac{m}{\rho} k \sin kL. \tag{8}$$

The problem of summing the solutions of this equation simplifies when the limit $L \to \infty$ is taken. We define phase shifts as follows:

$$k_n^\pm L = 2\pi n + 2\delta_n^\pm, \quad n = 0,1,2,\ldots, \tag{9}$$

and solve Eq. (8) in terms of $\delta_n^\pm$ as $L \to \infty$ for fixed k:

$$\delta_n^+ = 0, \qquad \delta_n^- = -\arctan[(m/\rho L)\pi n]. \tag{10}$$

Note that the unshifted momenta k_n^+ correspond to modes of oscillation which have a node at the location of the mass m on the string. The energy shift which results from adding the mass m is independent of the length L (for large L), and corresponds to a renormalization of the effective mass due to string fluctuations:

$$\delta m = -\frac{\hbar}{2\pi}\left(\frac{\kappa}{\rho}\right)^{1/2} \left[\Lambda \arctan\left(\frac{\Lambda m}{2\rho}\right) - \frac{\rho}{m}\ln\left\{1 + \left(\frac{\Lambda m}{2\rho}\right)^2\right\}\right](d-1). \tag{11}$$

When two masses are present ($N=2$), Eq. (6) becomes

$$0 = 2 - 2\cos kL + \frac{m_1+m_2}{\rho} k \sin kL - \frac{m_1 m_2}{\rho^2} k^2 \sin kx \sin k(L-x), \tag{12}$$

where $x = x_2 - x_1$. We parametrize the solutions again as in Eq. (9). One can show that the phase shifts $\delta_n^\pm$ are bounded in magnitude by $3\pi/2$ when m_1 and m_2 are varied from zero to infinity. Hence, in the limit $L \to \infty$ with k fixed, we can neglect δ_n compared with πn wherever k appears in Eq. (12) other than in the combination $\sin kL$ or $\cos kL$. Equation (12) is then quadratic in $\tan\delta_n$. It can be solved explicitly, and we find

$$\delta_n^+ + \delta_n^- = -\arctan\frac{\pi n(m_1+m_2)\rho L - \pi^2 n^2 m_1 m_2 \sin 4\pi n x/L}{\rho^2 L^2 - \pi^2 n^2 m_1 m_2 (1 - \cos 4\pi n x/L)}. \tag{13}$$

From the requirement that $\delta_n^+ + \delta_n^-$ be a continuous function of k_1, m_1, m_2, and x it follows that the arctan function takes values in the interval $[-\pi/2, \pi/2]$ when the denominator $\rho^2 L^2 - \pi^2 n^2 m_1 m_2 [1 - \cos(4\pi n x/L)]$ is positive, and in the interval $[\pi/2, 3\pi/2]$ when that denominator is negative.

Thus, the shift in zero point energy when any two masses are added to the string is given in the $L \to \infty$ limit by

$$\delta E = -(d-1)\frac{\hbar}{2\pi}\left(\frac{\kappa}{\rho}\right)^{1/2} \int_0^\infty dk\, f\left(\frac{k}{\Lambda}\right) \arctan\frac{2k(m_1+m_2)\rho - k^2 m_1 m_2 \sin 2kx}{4\rho^2 - k^2 m_1 m_2 (1 - \cos 2kx)}, \tag{14}$$

where $f(x)$ is a smooth function approximating the step function $\theta(1-x)$. In the limit of large k, the arctan function in Eq. (14) equals

$$\frac{3\pi}{2} - kx(\mathrm{mod}\pi) - \frac{1}{k}\frac{m_1+m_2}{m_1m_2}\rho + O\left(\frac{1}{k^2}\right).$$

It is clear from this limiting behavior that δE depends upon the cutoff Λ only through x-independent terms. Hence one of the main results of this paper: The interaction forces are universal, i.e., independent of L and Λ for large L and Λ. On dimensional grounds, we then have

$$\delta E = \delta E_0 + V(x), \quad V(x) = \frac{1}{x}F\left(\frac{m_1m_2}{\rho^2x^2}, \frac{m_2}{m_1}\right). \tag{15}$$

There is a set of interesting limits that can be taken in this expression.

The effect of small masses $m_1, m_2 \ll \rho x$ is easily evaluated by expanding Eq. (14) in a power series. We find, in this regime, a universal two-body interaction potential, proportional to the two masses m_1 and m_2 and falling off as the inverse cube of distance:

$$V(x) = -(d-1)\frac{\hbar}{32\pi}\frac{m_1m_2}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{1/2}\left[\frac{1}{x^3} - \frac{3}{4}\frac{m_1+m_2}{\rho}\frac{1}{x^4}\right] + O\left(\frac{m^2}{x^5}\right). \tag{16}$$

The effect of one small mass m to first order in that mass but all orders in the other mass M is also interesting and given by the following expression:

$$V(x) = \frac{\hbar}{2\pi}\left(\frac{\kappa}{\rho}\right)^{1/2}(d-1)m\int_0^\infty dk \frac{k^2M}{4\rho^2 + k^2M^2}\left(\frac{kM}{2\rho}\cos 2kx + \sin 2kx\right). \tag{17}$$

In the limit of small M, we recover Eq. (16), and in the limit of infinite M, we obtain

$$V(x) = -(d-1)\frac{\hbar m}{16\pi\rho}\left(\frac{\kappa}{\rho}\right)^{1/2}\frac{1}{x^2}. \tag{18}$$

In the limit where both masses are large, we have

$$V(x) = -(d-1)\hbar\left(\frac{\kappa}{\rho}\right)^{1/2}\left[\frac{\pi}{24x} - \frac{\rho(m_1+m_2)}{4\pi m_1m_2}\ln x + O\left(\frac{x\rho^2}{m^2}\right)\right]. \tag{19}$$

The $1/x$ leading part of this potential was derived originally in [7], where it was also shown to be universal. Equation (14) is a systematic expression for corrections to this leading behavior.

The equation for a general number N of additional masses can also be readily derived from Eq. (6), and reads (with $x_{ij} = x_j - x_i$):

$$0 = 2 - 2\cos kL - \sum_{p=1}^{N}\left(-\frac{k}{\rho}\right)^p \sum_{(\alpha_1\cdots\alpha_p)}\left(\prod_{i=1}^{p} m_{\alpha_i}\right)\left(\prod_{i=2}^{p}\sin kx_{\alpha_{i-1}\alpha_i}\right)\sin k(L - x_{\alpha_1\alpha_p}), \tag{20}$$

where the $(\alpha_1 \cdots \alpha_p)$ are all possible ordered $(\alpha_1 < \alpha_2 \cdots < \alpha_p)$ subsets with p elements of $(1,2,\ldots,N)$. This equation can also be solved in the limit where $L \to \infty$, and one finds

$$\delta_n = \delta_n^+ + \delta_n^- = \arctan\frac{a_{\cos}}{2 - a_{\sin}}, \tag{21}$$

where

$$a_{\binom{\cos}{\sin}} = \sum_{p=1}^{N}\left(-\frac{k}{\rho}\right)^p \sum_{(\alpha_1\cdots\alpha_p)}\left(\prod_{i=1}^{p} m_{\alpha_i}\right)\left(\prod_{i=2}^{p}\sin kx_{\alpha_{i-1}\alpha_i}\right)\begin{pmatrix}\cos\\ \sin\end{pmatrix}(kx_{\alpha_1\alpha_p}) \tag{22}$$

and the energy shift may be written down in analogy with Eq. (14). The three-body forces may be of special interest. They are given in the limit of small masses by

$$V(x_{12}, x_{13}, x_{23}) = +(d-1)\frac{3}{64\pi}\hbar\left(\frac{\kappa}{\rho}\right)^{1/2}\frac{m_1m_2m_3}{\rho^3}\frac{1}{x_{13}^4} \tag{23}$$

for three masses with ordering $x_1 < x_2 < x_3$. Note that two-body forces dominate over three-body and higher forces at large distances. One can also evaluate the three-body forces in the large mass (i.e., small distance) limit. It is found that the two-body forces dominate in that limit as well.

We conclude with some additional remarks about the nature of these Casimir type forces. The most interesting physical implication of our results is perhaps that all forces being attractive between all beads, an instability occurs along the lines of the instability present in gravity. Masses attached to a straight string tend to aggregate. The forces we have investigated are similar to gravity in other respects as well. Indeed we found that the force between two inertial masses m_1 and m_2, attached to a straight string, is proportional to the product $m_1 m_2$ in the limit of small masses. Therefore this force obeys the equivalence principle in that limit. One may even hope to explain the phenomenon of gravity by postulating that we and everything we know of are attached as zero (or near zero) modes to a flat 3+1 dimensional hypersheet embedded in a higher dimensional space. However, we found that the force between the two pointlike masses on the string, in the limit of small masses where the "principle of equivalence" holds, is proportional to $-m_1 m_2/\rho^2 r^4$, where r is the distance between the masses. Its naive generalization to 3+1 dimensions, $-m_1 m_2/\epsilon^2 r^8$, where ϵ is the energy per unit volume of the hypersheet, does not have the desired r dependence. It is tantalizing, however, that the equivalence of inertial and "gravitational" masses in the toy model arises naturally from quantum mechanical effects.

We would like to thank the Institute for Theoretical Physics at University of California, Santa Barbara and the Aspen Center for Physics for providing the forum where this work was initiated and for their hospitality. This research was supported in part by NSF Grant No. PHY-89-15286 and by DOE Contract No. DE-FG05-86ER40272.

*Electronic address: DHOKER@UCLAHEP.BITNET
†Electronic address: SIKIVIE@UFHEPA.PHYS.UFL.EDU

[1] Y. Nambu, in *Proceedings of the International Conference on Symmetries and Quark Models*, Wayne State University, 1969, edited by Ramesh Chand (Gordon and Breach, New York, 1970); T. Goto, Prog. Theor. Phys. **46**, 1560 (1971).

[2] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).

[3] A. M. Polyakov, Nucl. Phys. **B268**, 406 (1986).

[4] J. Polchinski and A. Strominger, Phys. Rev. Lett. **67**, 1681 (1991).

[5] J. Hong, J. Kim, and P. Sikivie, Phys. Rev. Lett. **69**, 2611 (1992).

[6] Open strings with masses attached at their ends were considered as elementary particle models in A. Chodos and C. B. Thorn, Nucl. Phys. **B72**, 509 (1974); I. Bars and A. J. Hanson, Phys. Rev. D **13**, 1744 (1976); W. A. Bardeen, I. Bars, A. J. Hanson, and R. D. Peccei, Phys. Rev. D **13**, 2364 (1976).

[7] M. Lüscher, Nucl. Phys. **B180**, 317 (1981).