Testing for the Gaussian Nature of Cosmological Density Perturbations through the Three-Point Temperature Correlation Function

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One of the crucial aspects of density perturbations that are produced by the standard inflation scenario is that they are Gaussian where seeds produced by topological defects tend to be non-Gaussian. The three-point correlation function of the temperature anisotropy of the cosmic microwave background radiation (CBR) provides a sensitive test of this aspect of the primordial density field. In this paper, this function is calculated in the general context of various allowed non-Gaussian models. It is shown that the Cosmic Background Explorer and the forthcoming South Pole and balloon CBR anisotropy data may be able to provide a crucial test of the Gaussian nature of the perturbations.

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Testing for the Gaussian nature of the primordial fluctuation spectrum is of critical importance to many cosmological models. In particular, traditional cosmic inflation [1] specifically predicts a Gaussian density fluctuation spectrum. The scale invariant quantum fluctuations generated during the inflationary epoch are expected to serve as the primordial density perturbations which develop into the large scale structures we observe today [2]. Competing models for structure formation, including topological defects originating from cosmological phase transitions [3] and nonstandard inflation models [4], will also generate a scale invariant (or nearly scale invariant) power spectrum for density perturbations similar to that of inflation. However, the statistics of these latter fluctuations are non-Gaussian. Thus, the Gaussian nature of the fluctuations provides a unique handle in discriminating different structure formation scenarios. In this Letter, we will discuss how to test this aspect of the primordial density field through the temperature anisotropy of the cosmic microwave background radiation (CBR).

As we showed [5], in momentum space, the lowest order deviation from Gaussian behavior is described by the bispectrum of the gravitational potential ϕ, $P_\phi(k_1,k_2,k_3)=\langle\phi_{k_1}\phi_{k_2}\phi_{k_3}\rangle$ $(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0.)$ When the perturbation is adiabatic so that the temperature anisotropy is related to the gravitational potential ϕ at the last scattering surface through the Sachs-Wolfe [6] formula,

$$\frac{\delta T}{T}=\frac{\phi}{3}\,, \tag{1}$$

the three-point temperature correlation function is related to the bispectrum through

$$\xi_T(\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\mathbf{l}})=\frac{1}{27}\int P_\phi(k_1,k_2,k_3)e^{i(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{m}}+i\hat{\mathbf{k}}_2\cdot\hat{\mathbf{n}}+i\hat{\mathbf{k}}_3\cdot\hat{\mathbf{l}})\eta_0}\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\,, \tag{2}$$

where $\eta_0=2H_0^{-1}$ is the distance to the last scattering surface (we set $\eta_0=1$ thereafter), and $\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\mathbf{l}}$ are the beam directions. A nonvanishing three-point function clearly indicates that the bispectrum is not zero. Note that for Gaussian primordial perturbations, the bispectrum is strictly zero in all cases. Thus, the three-point temperature correlation function is a clean test of the Gaussian character of primordial fluctuations.

In [7], Falk, Rangarajan, and Srednicki found that in an inflationary model with cubic self-interaction, $P_\phi(k_1,k_2,k_3)$ is given by

$$P_\phi=\beta(k_1k_2k_3)^{-3}(k_1^3+k_2^3+k_3^3)\,, \tag{3}$$

where $\beta\sim10^{-6}$. In this paper, we will show that without invoking any new assumptions about inflationary models, the nonlinear gravitational evolution of the initial Gaussian perturbations will give rise to a three-point correlation function, which has a similar angular dependence to that which certain non-Gaussian inflationary models predict, but a much larger amplitude than the one considered in [7]. Then, we extend the analysis in [7] to include more general cases of inflation, which produce not only the scale invariant but also the "tilted" perturbation spectrum [8]. The extended analysis is helpful in discussing the effect of the spectral index on the angular dependence of the three-point function. To choose different non-Gaussian inflationary models through three-point temperature correlation function will be hard because of the gravitational evolutionary effects. However, the three-point correlation function produced by a cosmological phase transition tends to have a distinctive angular dependence, which should enable one to prove or disprove the scenario through observations. Finally, we briefly discuss the effect of noise in the sky signal of CBR measurements on the analysis of the three-point temperature correlation function.

By taking into account nonlinear gravitational evolution, it is found that there are two terms which contribute

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to the CBR temperature anisotropy

$$\frac{\delta T}{T}=\frac{\phi}{3}+2\int\frac{\partial\phi}{\partial\eta}d\eta\,, \tag{4}$$

where the first term is the Sachs-Wolfe [6] effect due to the gravitational potential at the last scattering surface, and the second term is the generalized Rees-Sciama effect [9] due the evolution of the gravitational potential along the photon path. When we adopt a flat cosmological model ($\Omega=1$), the quasinonlinear analysis gives [5,10,11]

$$\phi(k,\eta)=\phi_i(k)+\left(\frac{a(\eta)}{14\times 36}\right)\int J(\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}')k^2 \times\phi_i(k')\phi_i(k-k')\,, \tag{5}$$

where ϕ_i is the gravitational potential at the last scattering surface, $a(\eta)=(\eta/\eta_i)^2$ is the expansion factor of the Universe after decoupling, and

$$J(\mathbf{k},\mathbf{l},\mathbf{m})=2(\mathbf{l}\cdot\mathbf{m})+\frac{5(\mathbf{k}\cdot\mathbf{l})m^2}{k^2}+\frac{5(\mathbf{k}\cdot\mathbf{m})l^2}{k^2}\,. \tag{6}$$

We first estimate the amplitude of the second term relative to the first term: Since the expansion factor a after decoupling is $\sim(1+z_{\rm dec})\sim 1000$, the amplitude of the gravitational potential at the last scattering surface is around 10^{-5} as suggested by the measurement by the Cosmic Background Explorer (COBE); thus the ratio of the Rees-Sciama term to the Sachs-Wolfe term is of order 0.01–0.1. As the nonlinear effects are contained in the Rees-Sciama term, it is this term that contributes significantly to the three-point correlation function. For comparison, the nonlinear term considered in [7] is 10^{-6} times smaller than the linear term. This is a potential problem of testing inflationary models through the three-point temperature correlation function. To be observable the amplitude of the non-Gaussianity produced in these models has to be large enough so that the gravitational evolution cannot completely dominate.

Although the Rees-Sciama contribution to the three-point function will overwhelm the inflationary contribution in the model considered in [7], it is not true generally. The generic form of the bispectrum in inflationary models is given by

$$P_\phi(k_1,k_2,k_3)=\lambda[P_\phi(k_1)P_\phi(k_2)+P_\phi(k_2)P_\phi(k_3) +P_\phi(k_3)P_\phi(k_1)]\,, \tag{7}$$

where λ is a constant and $P_\phi=\langle\phi(k)\phi(-k)\rangle$ is related to the power spectrum of density perturbation $P(k)$ simply through $P_\phi(k)=36P(k)k^{-4}$. The cubic self-interaction model corresponds to the case where $\lambda\sim 10^{-6}$ with a scale invariant density perturbation spectrum, or equivalently, $P_\phi(k)\sim k^{-3}$; the nonlinear gravitational evolution effect corresponds to the case where $\lambda\sim 2(1+z_{\rm dec})/9^3\sim 2.0$ and $P_\phi(k)\approx k^{-2}$.

Note that the two-point temperature correlation function is related to P_ϕ through

$$C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})=\frac{1}{9}\int P_\phi(k)e^{i\mathbf{k}\cdot(\hat{\mathbf{m}}-\hat{\mathbf{n}})\eta_0}\frac{d^3k}{(2\pi)^3}\,. \tag{8}$$

The three-point correlation function calculated from the bispectrum given by Eq. (7) is

$$\xi_T(\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\mathbf{l}})=3\lambda[C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{n}},\hat{\mathbf{l}})+C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{m}},\hat{\mathbf{l}}) +C_2(\hat{\mathbf{m}},\hat{\mathbf{l}})C_2(\hat{\mathbf{n}},\hat{\mathbf{l}})]\,. \tag{9}$$

This is a theoretical relation between three-point and two-point functions since the finite-beam size effects have not been taken into account yet. The formal treatment of the finite-beam effect in the CBR experiment can be found in [12,13]. The beam can be well approximated as a Gaussian,

$$f(|\hat{\mathbf{m}}-\hat{\mathbf{n}}|,\sigma)=\frac{1}{2\pi\sigma^2}e^{-|\hat{\mathbf{m}}-\hat{\mathbf{n}}|^2/2\sigma^2}\,, \tag{10}$$

and the observed temperature correlation function will be the convolution of the theoretical correlation (infinite thin beam) with the beam, which is

$$C_3(\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\mathbf{l}},\sigma)=\int d\Omega_1' d\Omega_2' d\Omega_3' f(|\hat{\mathbf{m}}-\hat{\mathbf{m}}'|,\sigma)f(|\hat{\mathbf{n}}-\hat{\mathbf{n}}'|,\sigma)f(|\hat{\mathbf{l}}-\hat{\mathbf{l}}'|,\sigma)C_3(\hat{\mathbf{m}}',\hat{\mathbf{n}}',\hat{\mathbf{k}},0)\,. \tag{11}$$

For a special configuration of three beams where $\hat{\mathbf{m}}\cdot\hat{\mathbf{n}}=\hat{\mathbf{n}}\cdot\hat{\mathbf{l}}=\hat{\mathbf{l}}\cdot\hat{\mathbf{m}}=\cos\alpha$, the beam-smoothed three-point function is well approximated [7] as $[C_2(\cos\alpha|\sigma)]^2$ where $C_2(\cos\alpha|\sigma)$ is the two-point function with the monopole, dipole, and quadrupole terms removed. Since the three-point function is the product of two-point functions, it has a stronger dependence on the power spectra index n. Multipole expansion of the two-point function gives

$$C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})=\sum_l C_l(2l+1)P_l(\hat{\mathbf{m}}\cdot\hat{\mathbf{n}})\,. \tag{12}$$

For a power law spectrum $P(k)\sim k^n$, C_l is given by

$$C_l=\frac{1}{5}\left(\frac{Q_{\rm rms}}{T_0}\right)^2\frac{\Gamma(2l+n-1)}{\Gamma(2l+5-n)}\frac{\Gamma[(9-n)/2]}{\Gamma[(3-n)/2]}\,, \tag{13}$$

where $Q_{\rm rms}$ is the COBE-measured quadrupole [14] and T_0 is the black-body temperature of CBR. From the analysis of the two-point correlation function, COBE can only put a loose bound on the power spectra index [14]: $n=1.1\pm0.5$. In Fig. 1, we plot the three-point function for two different power spectra: a scale invariant $n=1$ spectrum and a "tilted" spectrum where $n=0.7$. Notice how the three-point function depends strongly upon the power spectra index. Thus it is anticipated that the analysis of the three-point correlation function will put a more stringent bound on n.

In order to test cosmological structure formation scenarios through the three-point temperature correlation function, we should have a clear handle on what various

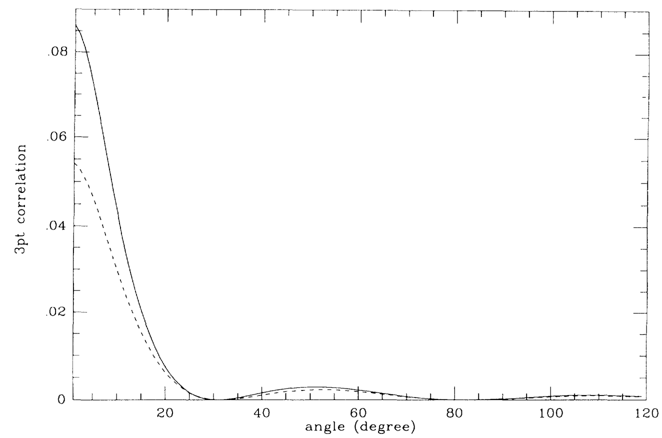

FIG. 1. The dependence of the three-point temperature correlation on the power spectra index and angular separation. The solid line is for a scale invariant density perturbation; the dashed line is for a "tilted" spectrum with spectra index $=0.7$.

models predict. In the following, we will show that the cosmological phase transition can produce distinctive angular dependences other than the form given above.

Cosmological phase transitions are widely discussed in the context of structure formation [15]. In the case of a primordial phase transition, the horizon size at the epoch of phase transition is small and topological defects will form according to the Kibble mechanism [16]. The analysis of the three-point correlation for defect-induced temperature anisotropy depends crucially on the evolution of the defect network and the work along this line is still in progress. In this paper, we will show that it is instructive to consider initially the three-point function in the late-time phase transition (LTPT) scenario [17]. The calculation is considerably simplified in LTPT models for the following reasons: (1) The last scattering surface is assumed smooth in LTPT models; thus temperature anisotropies are solely produced by the generalized Rees-Sciama effect,

$$\frac{\delta T}{T}=2\int\frac{\partial\phi}{\partial\eta}d\eta\,, \tag{14}$$

since the fluctuations in density and gravitational potential are generated by the critical fluctuations at the critical point of the phase transition, $\partial\phi/\partial\eta=\phi\delta(\eta-\eta_p)$, where η_p is the conformal time at the phase transition point. Thus, in this late-time phase transition model, the temperature anisotropy takes the following simple form:

$$\delta T/T=2\phi_{\eta=\eta_p}\,. \tag{15}$$

(2) For LTPT, the horizon size is large so that the finite horizon-size effect is negligible. We can calculate the three-point correlation function from symmetry considerations. As pointed out by Polyakov [18], the three-point correlation function of the fluctuating field ψ is completely determined up to a dimensionless constant by the conformal symmetry of the system at the critical point. The explicit form for the three-point function is given by

$$\xi_3=\langle\psi(x_1)\psi(x_2)\psi(x_3)\rangle$$
$$=\eta c_2(x_1,x_2)c_2(x_2,x_3)c_2(x_3,x_1)\,, \tag{16}$$

where $c_2(x_1,x_2)=\langle\psi(x_1)\psi(x_2)\rangle$ is the two-point function and η is a constant. In this Letter, we assume that the gravitational potential ϕ is directly proportional to the underlying fluctuating field ψ. For this case, the three-point temperature correlation function has the following simple relation to the two-point function:

$$\xi_T(\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\boldsymbol{l}})=A\cdot C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{n}},\hat{\boldsymbol{l}})C_2(\hat{\mathbf{m}},\hat{\boldsymbol{l}})\,, \tag{17}$$

where A is a dimensionless constant. The full beam-smearing effects of the three-point correlation function given above are messy and we will report them elsewhere. However, in the special case when $\hat{\mathbf{m}}\cdot\hat{\mathbf{n}}=\hat{\mathbf{n}}\cdot\hat{\boldsymbol{l}}=\hat{\boldsymbol{l}}\cdot\hat{\mathbf{m}}=\cos\alpha$, it can be approximated as $[C_2(\cos\alpha|\sigma)]^3$, where $C_2(\cos\alpha|\sigma)$ is the two-point function with finite-beam width σ, with monopole, dipole, and quadrupole terms subtracted.

The results obtained from Eqs. (9) and (13) strongly suggest that the general form of the three-point function, expressed in terms of two-point functions, is given by

$$\xi_T(\hat{\mathbf{m}},\hat{\mathbf{n}},\hat{\boldsymbol{l}})=Q[C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{n}},\hat{\boldsymbol{l}})+C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{m}},\hat{\boldsymbol{l}})+C_2(\hat{\mathbf{m}},\hat{\boldsymbol{l}})C_2(\hat{\mathbf{n}},\hat{\boldsymbol{l}})]+A\cdot C_2(\hat{\mathbf{m}},\hat{\mathbf{n}})C_2(\hat{\mathbf{n}},\hat{\boldsymbol{l}})C_2(\hat{\mathbf{m}},\hat{\boldsymbol{l}})\,, \tag{18}$$

where Q and A are constants. This is the archetype form of the three-point correlation function that the experimental analysis should be compared with.

The Gaussian character can be tested through the existing COBE and the forthcoming South Pole and balloon CBR anisotropy data by the three-point temperature correlation function. Tests for Gaussian behavior are interesting and timely since the question is still unresolved. For example, the recent MAX balloon experiments [19], the new thirteen-points scan from the South Pole [20], and the MSAM/GSFC balloon experiment [21] all seem to indicate that on degree scales the sky CBR signals are possibly not Gaussian distributed. Detailed statistical analysis of the South Pole data set shows that the quality of fit to non-Gaussian distributions is superior to the quality of Gaussian assumptions for this data set [22]. Although at the present stage the experiments are inconclusive due to possible foreground contaminations, nevertheless they give us hope that the question of Gaussian behavior might be resolved experimentally in the near future. In this Letter, we focus on the COBE data although the idea and method discussed can equally apply to the South Pole and balloon experiments. The data set from the COBE Differential Microwave Radiometers should be suitable for carrying out this test. On the one hand, the beam width of COBE is 7°, which is much larger than the horizon size at decoupling ($\sim 2°$). Most nonlinear causal processes which may lead to non-Gaussian signatures on the cosmic microwave background sky are smoothed out by the beam. On the other hand, the COBE CMB map covers the whole sky. Thus, the boundary effects will be minimized. However, the detected sky signal contains both the intrinsic CBR temperature fluctuation and the instrumental noises,

$$\left(\frac{\delta T}{T}\right)_{\mathrm{obs}} = \left(\frac{\delta T}{T}\right)_{\mathrm{CBR}} + \left(\frac{\delta T}{T}\right)_{\mathrm{noise}} . \tag{19}$$

The signal to noise ratio of the COBE data is around 1:1 and this is typical in all current CBR temperature anisotropy experiments. Thus, it is important to consider the noise term seriously in the analysis of the three-point correlation function. Even if future analysis of the COBE data does find a nonvanishing three-point temperature correlation, it may be due to the instrumental noise. However, if one adopts the usual assumptions about the noise term, i.e., (1) the noise is random Gaussian noise which is not correlated temporally or spatially, and (2) the noise is not correlated with the CBR signal, then

$$\xi_{\mathrm{obs}} = \xi_{\mathrm{CBR}} ; \tag{20}$$

the three-point correlation calculated from the raw observational data will reflect directly the three-point temperature correlation of the CBR, even if the noise term is comparable to the signal. This is another advantage in using the three-point function to test for Gaussian initial perturbations.

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[1] A. Guth, Phys. Rev. D **23**, 347 (1981); A. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[2] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983); A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinskii, Phys. Lett. **117B**, 175 (1982).

[3] For a review, see A. Vilenkin, Phys. Rep. **121**, 263 (1985); N. Turok, Phys. Rev. Lett. **63**, 2625 (1989); N. Turok and D. N. Spergel, Phys. Rev. Lett. **66**, 3093 (1991).

[4] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990); D. S. Salopek, J. R. Bond, and G. Efstathiou, Phys. Rev. D **40**, 1753 (1989); H. M. Hodges, G. R. Blumenthal, L. A. Koffman, and J. R. Primack, Nucl. Phys. **B335**, 197 (1990).

[5] X. Luo and D. N. Schramm, Astrophys. J. **408**, 33 (1993).

[6] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).

[7] T. Falk, R. Rangarajan, and M. Srednicki, Astrophys. J. **403**, L1 (1993).

[8] K. Freese, J. Frieman, and A. Olinto, Phys. Rev. Lett. **65**, 3233 (1990); F. Adams *et al.*, Phys. Rev. D **47**, 426 (1993); A. Liddle and D. Lyth, Phys. Rep. (to be published); R. Cen *et al.* Astrophys. J. **399**, L11 (1992).

[9] M. Rees and D. Sciama, Nature (London) **217**, 511 (1968); E. Martinez-Gonzalez, J. L. Sanz, and J. Silk, Phys. Rev. D **46**, 4193 (1992); A. Jaffe, A. Stebbins, and J. Frieman, Report No. Fermilab-Pub-92/362-A, 1992 (to be published).

[10] R. Juszkiewicz, Mon. Not. R. Astron. Soc. **197**, 931 (1981).

[11] E. T. Vishniac, Mon. Not. R. Astron. Soc. **203**, 345 (1983).

[12] M. L. Wilson and J. Silk, Astrophys. J. **243**, 14 (1981).

[13] N. Gouda, M. Sasaki, and Y. Suto, Astrophys. J. **341**, 557 (1989).

[14] G. Smoot *et al.*, Astrophys. J. **396**, L1 (1992); E. Wright *et al.*, Astrophys. J. **396**, L13 (1992).

[15] E. Kolb and M. S. Turner, *The Early Universe* (San Francisco, Addison-Wesley, 1989).

[16] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).

[17] C. T. Hill, D. N. Schramm, and J. Fry, Comments Nucl. Part. Phys. **19**, 25 (1989); X. Luo and D. N. Schramm, Report No. Fermilab-Pub-93/020-A, 1993 (to be published); E. Kolb and Y. Wang, Phys. Rev. D **45**, 4421 (1992).

[18] A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **538** (1970) [JETP Lett. **12**, 381 (1970)].

[19] J. O. Gunderson *et al.*, CfPA report, 1993 (to be published); P. R. Meinhold *et al.*, CfPA report, 1993 (to be published).

[20] J. Schuster *et al.*, Santa Barbara report, 1993 (to be published).

[21] E. S. Cheng *et al.*, NASA/GSFC report, 1993 (to be published).

[22] P. Graham, N. Turok, P. M. Lubin, and J. A. Schuster, Report No. PUP-TH-1408, 1993 (to be published).