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Effective Wave Number for Transmission of Linear Waves in One-Dimensional Media

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An approximate wave number formula is derived for one-dimensional wave transmission in linear systems that are uniform on a large scale. It is a low-frequency approximation that coincides at zero frequency with equivalent medium theory. The formula is evaluated for various lattice media models for which exact calculation can also be made.

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Waves propagating in one-dimensional nonuniform systems have been studied widely, both theoretically and through simulation and experiment. Many different one-dimensional systems are modeled by very similar wave equations, systems such as compressional wave propagation in a stratified elastic medium [1], extensional and torsional waves in elastic rods [2], acoustic waves in fluid-filled ducts [3], and surface waves in fluids [4–6]. Of special interest are systems which appear uniform at a large scale, but vary at a small scale. Examples of these are periodic systems or statistically homogeneous systems but many other types are possible. Wave transmission in such systems is qualitatively similar to the transmission of waves in uniform dispersive systems. In the nonuniform system the apparent attenuation and dispersion is caused by the scattering of the wave from inhomogeneities which both reduces the amplitude of the wave and delays its arrival. The apparent attenuation of transmitted waves due to scattering has long been known in seismology [7], as has the close connection between apparent attenuation and the theory of wave-function localization as first noted in [8]. Many calculations of apparent attenuation and wave number have been made, but generally these apply to restricted types of systems, or are exact only in the limit of small variations in the system parameters [9–12]. The method outlined in this Letter is exact in the low-frequency limit and is applied here to the perfectly elastic wave equation. However, the addition of attenuation or differing wave speeds for forward- and backward-going waves is trivial (an example of the latter is one-dimensional acoustics in a fluid-filled duct with a mean flow).

The calculations in this Letter are based on the one-dimensional elastic wave equation, which will be written in the form

$$\frac{\partial}{\partial x} f = M(x)\frac{\partial}{\partial t} f \, , \tag{1}$$

where f is the stress and velocity vector

$$f = \begin{pmatrix} F \\ v \end{pmatrix} \, , \tag{2}$$

and M is the elastic matrix, comprising a density (ρ) and a compliance (λ),

$$M = \begin{pmatrix} 0 & \rho \\ \lambda & 0 \end{pmatrix} \, . \tag{3}$$

Fourier transforming in time leads to the ordinary differential equation

$$\frac{df}{dx} = -i\omega M(x) f \, , \tag{4}$$

where ω is the angular frequency.

The integral equation from which the effective wave number derives comes from considering wave amplitudes, which belong to the dual of the solution space of Eq. (4). The dual equation (also in this case the adjoint equation) is

$$\frac{df^\dagger}{dx} = i\omega f^\dagger M(x) \, , \tag{5}$$

where $f^\dagger$ is a row vector.

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Consider another infinite one-dimensional material, with coordinate y, and two locations in the original material, a and x with $a < x$. The elastic matrix of the new material is defined as being equal to M for $a \leq y \leq x$. Outside of the interval $[a, x]$ it is not necessary to define the elastic matrix fully, only that the impedance has the constant value of ζ so that the material supports conventional propagating waves. Transmission and reflection of waves through the zone of varying impedance depends on the two locations a and x. If a unit amplitude wave is incident from $y < a$, the transmitted wave in $y > x$ is defined to have amplitude $1/S(x,a)$. If a unit amplitude wave is incident from $y > x$, the reflected wave in $y > x$ is defined to be of amplitude $R(x,a)$.

It may be shown that S and $-SR$ are the components of a solution to Eq. (5) in a basis that depends on ζ. Writing

$$f^{\dagger} = (S - SR) \ , \tag{6}$$

then

$$\frac{d}{dx}(S - SR) = i\omega (S - SR) \begin{pmatrix} -\sigma & \mu \\ -\mu & \sigma \end{pmatrix} , \tag{7}$$

where

$$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{\zeta} & \zeta \\ \frac{1}{\zeta} & -\zeta \end{pmatrix} \begin{pmatrix} \rho \\ \lambda \end{pmatrix} . \tag{8}$$

The boundary conditions for S and R are

$$S(a,a) = 1, \quad R(a,a) = 0. \tag{9}$$

Rather than work directly with S it is convenient to use a function Q phase shifted by σ,

$$Q(x,a) = \exp\left(i\omega \int_a^x \sigma(u) du\right) S(x,a) \ , \tag{10}$$

and convert Eq. (7) into an integral equation, giving

$$Q(x,a) = 1 + \omega^2 \int_{z=a}^{z=x} Q(z,a)\mu(z) \times \int_{y=z}^{y=x} \mu(y) \exp\left(2i\omega \int_z^y \sigma(u) du\right) dy\, dz \ . \tag{11}$$

Equation (11) is exact, no approximations having been made. It is very similar to the integral equation from which the Bremmer series may be derived. While the Bremmer series arises from a decomposition into locally defined forward- and backward-going waves, the decomposition in (11) is with respect to a fixed basis. An important consequence of this is that while the Bremmer series involves the spatial derivative of the impedance, Eq. (11) makes no assumptions about the differentiability of ρ and λ. Equation (11) applies for systems where the mean value of M is diagonalizable but $M(x)$ is not diagonalizable for any x, such as mechanical systems of massless springs and rigid masses.

By allowing ζ to vary spatially Eq. (7) may be generalized to include terms in the derivative of ζ. If ζ is chosen to be $(\rho/\lambda)^{\frac{1}{2}}$ then the Bremmer series equation results [13]. Recent work in dissipative wave equations also uses a decomposition that is not the local forward- and backward-going wave decomposition. The basis used is defined locally in terms of the nondissipative part of the full equation [14,15], or is matched to a local basis with constant dissipation [16].

The advantage of using Eq. (11) to derive a low-frequency effective wave number is that if ρ and λ are bounded, then $\omega\mu$ will be small for sufficiently small ω. To derive an effective wave number an approximate solution will be found to (11) when the material parameters ρ and λ are approximately spatially invariant. In that case $Q(x,a)$ should approximately depend only on $x - a$.

The finite y integral in (11) can be written as the difference between two infinite integrals:

$$Q(x+a,a) = 1 + \omega^2 \int_{z=a}^{z=x+a} Q(z,a) \left\{ I(z,z) - \exp\left(2i\omega \int_z^{x+a} \sigma(u) du\right) I(z, x+a) \right\} dz \ , \tag{12}$$

where

$$I(u,v) = \int_{y=v}^{y=\infty} \mu(u)\mu(y) \exp\left(2i\omega \int_v^y \sigma(w) dw\right) dy \ . \tag{13}$$

So long as $I(z,z)$ exists, $I(z, x+a) - I(z,z)$ will be bounded, since ρ and λ are approximately invariant. Also $I(z,z)$ will be approximately equal to its average value. If the exponential term in (12) is approximated using an averaged σ, then

$$Q(x+a,a) \approx 1 + \omega^2 \int_a^{x+a} Q(z,a) \times \{1 - \exp(2i\omega\bar{\sigma}[x+a-z])\} I\, dz \ , \tag{14}$$

where

$$I = \langle I(z,z) \rangle , \quad \bar{\sigma} = \langle \sigma \rangle \ , \tag{15}$$

the solution to which is

$$Q(x+a,a) \approx \exp(i\omega\bar{\sigma}x) \left(\cos(kx) - \frac{i\omega\bar{\sigma}}{k}\sin(kx)\right) , \tag{16}$$

where

$$k = \omega\bar{\sigma}\left(1 + \frac{2i\omega I}{\bar{\sigma}}\right)^{\frac{1}{2}} . \tag{17}$$

Converting back to the function $S(x,a)$ gives

$$S(x,a) \approx \left(\cos(k[x-a]) - \frac{i\omega\bar{\sigma}}{k}\sin(k[x-a])\right) . \tag{18}$$

Equation (18) has exactly the form for the inverse transmission function of a spatially constant (but dispersive) material between a and x, sandwiched between two half-spaces of a different material. It is reasonable therefore to refer to k as the effective wave number for transmission. (A more sophisticated argument shows that a slightly more accurate value of $S(x,a)$ comes from replacing the globally averaged $\bar{\sigma}$ outside the square root in Eq. (17) by the average of σ over the interval $[a,x]$.)

The effective wave number k is not uniquely defined by Eq. (17), since it depends on the choice of ζ; however, in two limiting cases k is independent of ζ. Equation (14) is exact if ρ and λ are constant, even when $\mu \neq 0$. It may also be checked that in the low-frequency limit k tends to the wave number of the averaged medium, given by

$$k = (\langle\rho\rangle\,\langle\lambda\rangle)^{\frac{1}{2}} \tag{19}$$

for all ζ. It is clear therefore that k depends only weakly on ζ, although, of course, in general values of k for different values of ζ diverge from one another as the frequency increases.

In addition, Eq. (17) is in qualitative agreement with exact theory for periodic systems. If the material is periodic, then I is real, and so k is either pure real or pure imaginary. The wave numbers of Bloch modes in periodic elastic systems may be calculated exactly, and are also either pure real or pure imaginary [17,18].

There are two conditions under which Eq. (14) may not be expected to approximate the exact Eq. (11). First as the frequency increases, $\omega^2 I(z,x)$ diverges increasingly from $\omega^2\langle I(z,z)\rangle$. The error is small so long as over all intervals

$$\omega\left|\int[\mu(x)-\langle\mu\rangle]dx\right| \ll 1\,. \tag{20}$$

If ρ and λ are bounded then for any ζ there will be an ω below which condition (20) is satisfied. The second condition is when $I(z,z)$ does not exist. This condition will be returned to in the examples.

Although for transmission an approximation to the true transmission has been derived that has a similar form to transmission through a uniform material, the same is not true of reflection. Integrating Eq. (7) gives

$$R(x,a) \approx \frac{i\omega}{S(x,a)}\int_a^x \exp\left(i\omega\int_z^x \sigma(u)du\right)\mu(z)S(z,a)dz\,. \tag{21}$$

The formula for R has the appearance of a single-scattering approximation, but in the medium with wave number given by (17).

In the experiments of Kono and Nakade [5] one-dimensional structures were produced from elements of two types. The two types are of equal length but differed in impedance and sound speed. These elements were assembled in four types of arrangements, namely, periodically, randomly, and according to the rules of Fibonacci and Thue-Morse lattices. I shall show comparisons in the frequency and time domains between exact calculations and Eq. (17) for three of these arrangements (periodic, random, and Thue-Morse), with parameters chosen so as to give results similar to those of the experiments. The two elements have the same length (normalized to 1), and one is chosen to have sound speed and impedance of 1 [and hence in terms of the parameters of Eq. (3), $\rho=\lambda=1$]. The other element has parameters

$$\rho = 1.66, \quad \lambda = 0.81\,. \tag{22}$$

(These parameters reproduce the conditions of Fig. 1 and trace a of Fig. 2 of [5]. In the terminology of [5]

$$N = 1.16, \quad r = 1.43\,. \tag{23}$$

The value of r given in Table I of [5] is inconsistent with the observed transmission and phase velocity. That given here has the same zero-frequency phase velocity as trace a of Fig. 3 of [5].) In the calculation, ζ has been chosen as the impedance of the averaged medium, and so μ is ± 0.19. Integrating over one element of the array, the left-hand side of (20) is equal to 1 for $\omega \approx 5$.

There are two aspects to comparing exact and approximate calculations of transmission; amplitude and phase. While amplitude is easiest to compare in the frequency domain, phase can be compared best by examining the propagation of a pulse in the time domain.

Figure 1 shows the calculated transmission amplitude through 100 elements in the three arrangements. For the random arrangement, the logarithmic mean has been taken over five realizations. Figure 2 shows the same calculation using the approximate wave number calculated using Eq. (17). The features in the two figures agree closely, although the positions of amplitude troughs di-

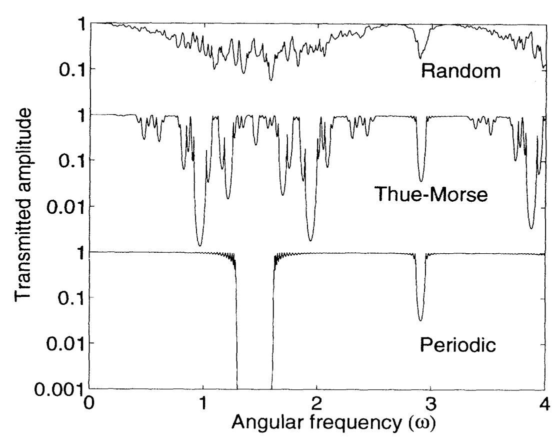

FIG. 1. An exact calculation of amplitude versus angular frequency for transmission through 100 elements (amplitudes have been cut off at 0.001).

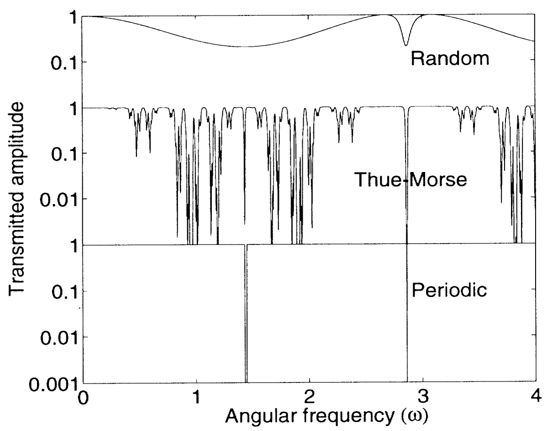

FIG. 2. A calculation of amplitude versus angular frequency for transmission through 100 elements using Eq. (17) (amplitudes have been cut off at 0.001).

verge as frequency increases. The most noticeable difference is in the width of the transmission gaps at approximately $\omega = \pi/2, \pi$. This is because the semi-infinite integral defining $I(z,z)$ diverges, as alluded to above. In the time domain this difference is only seen some time after the arrival of a pulse.

Figure 3 compares the propagation of a unit-height Gaussian pulse through the three arrangements. The original pulse had a $1/e$ width in the time domain of 1, in the angular frequency domain of 2. For each pair of traces the lower trace shows an exact calculation; the upper trace shows the calculation using the approximate wave number. Five traces have been averaged for the pulse through a random arrangement. The zero time for the figure is the time a pulse would take through a uniform medium with the same averaged density and compliance. The agreement is very good in all three cases up until 20 time units after the main arrival time. After this time the random arrangement shows a tail with uncorrelated variation. These tails differ for differing realizations of the random distribution. In contrast the Thue-Morse and periodic arrangements have tails that agree closely for different choices of 100 elements out of the infinite lattice distributions. For the periodic case, the tail is qualitatively similar in the exact and approximate calculations (a modulated wave), but with slightly different frequencies. For a periodic distribution an exact calculation of group velocity can be made, and this compared with the approximate formula. Figure 4 shows the exact group velocity (solid line) and the approximated group velocity (dashed line), with normalization by the group velocity of the averaged medium. For the first transmission gap there is only appreciable difference at velocities 20% below the zero-frequency velocity, so these are the components arriving 25 time units after the first arrival in Fig. 3. The dispersion relations are, however, qualitatively very similar [the similarity becomes more apparent if some intrinsic attenuation or slight disorder is added, since $I(z,z)$ will then always exist].

The derivation in this Letter has been restricted to the one-dimensional elastic wave equation for which the vector f has two elements. The extension to equations for which f has more than two elements introduces some complications, and is in general laborious (if f has four elements then semi-infinite double, triple, and quadruple correlation integrals are necessary, and effective wave numbers must be derived that correspond to submatrices of M.) Work on this will be reported elsewhere.

This Letter has benefited from discussions with col-

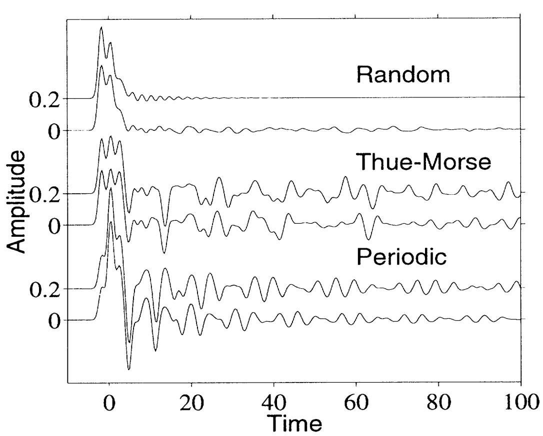

FIG. 3. Propagation of a Gaussian pulse through 100 elements as predicted using exact calculation (lower line of each pair) and Eq. (17) (lower line). The time is synchronized to the arrival time of a pulse in the averaged medium.

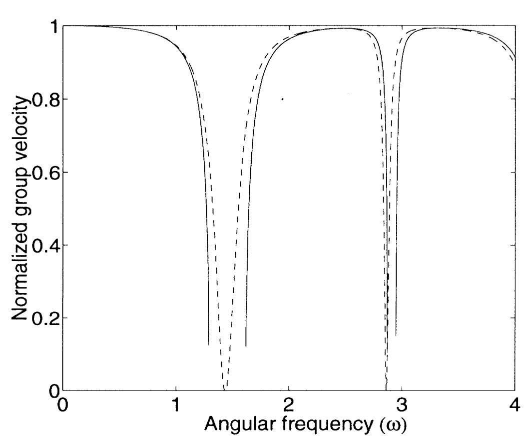

FIG. 4. A comparison of phase velocity calculated for the periodic arrangement using exact calculation (solid line) and Eq. (17) (dashed line). Velocities are normalized to the velocity of the averaged medium.

leagues, especially C. Chapman and M. de Hoop.

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