Nonanalytic Contributions to the Self-Energy and the Thermodynamics of Two-Dimensional Fermi Liquids

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We calculate the entropy of a two-dimensional Fermi liquid using a model with a contact interaction between fermions. We find that there are T^2 contributions to the entropy from interactions separate from those due to the collective modes. These T^2 contributions arise from nonanalytic corrections to the real part of the self-energy which may be calculated from the leading-log dependence of the imaginary part of the self-energy through the Kramers-Kronig relation. We find no evidence of a breakdown in Fermi liquid theory in 2D and conclude that Ferrmi liquids in 2D are similar to 3D Fermi liquids.

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The unusual nature of the normal state properties of the high temperature superconductors (HTS) has generated a new interest in the metallic phase of strongly correlated electronic materials. In particular, much attention has been focused on the existence [1–3] or nonexistence [4–8] of a Fermi liquid phase for these systems in two dimensions (2D). This controversy as to the existence of 2D Fermi liquid (FL) is motivated by the difficulty of fitting some experimental data on the HTS materials with conventional FL expressions and also by the property of one-dimensional systems that the ground state of a system of interacting fermions is a Luttinger liquid (LL) rather than a FL. In particular, this has led to the development of the marginal Fermi liquid (MFL) phenomenology [5,7] which has been used extensively to fit data [9]. However, there is no microscopic calculation as yet which leads to a MFL ground state.

The stability of a FL ground state has been studied extensively in the dilute limit and for weak coupling [3] in 2D. In the dilute limit it is possible to show that the particle-particle channel diagrams contribute in leading order. The presence of a two-hole bound state in this channel led to speculations that this was a possible source of the breakdown of the Fermi liquid phase [10]. It appears, though, that this bound state only gives rise to higher order corrections to the properties of the FL. In weak coupling away from half filling this stability of the FL phase of the 2D Hubbard model was also observed in the propagator renormalized fluctuation exchange approximation of Serene and Hess [3]. In this approach all of the known instabilities, superconductivity, spin and charge density waves, and the two-hole bound state could occur. No evidence for a breakdown of the FL phase was observed at quarter filling.

From these studies we see that the FL phase of the Hubbard model is stable against particle-hole or particle-particle fluctuations away from half filling. From the two-hole bound state it was shown [1] that this contributed a term of the order $|\epsilon|^{5/2}$ to the imaginary part of the self-energy, $\Sigma(p,\epsilon)$, and from Kramers-Kronig a similar term is found in $\mathrm{Re}\Sigma(p,\epsilon)$. The two-hole bound states are predominantly short wavelength fluctuations; in this Letter we investigate the long wavelength fluctuations. We find that they give rise to lower order nonanalytic corrections to FL theory than does the two-hole bound state. In particular, we find that $\mathrm{Re}\delta\Sigma(p,\xi_p) \propto \mathrm{sgn}(\xi_p)\xi_p^2$ and that this term gives rise to a T^2 correction to the specific heat, C_V. So the long wavelength effects which are responsible for the leading corrections in 3D also lead to the leading corrections in 2D. Bound state effects lead to higher order contributions. The nature of the leading corrections to the specific heat in 2D, $\sim T^2$, are such that are easily obscured by the contribution from collective modes but are presumably present in the results of Ref. [3]. What we learn from our work and that of Ref. [1] and [2] is that the breakdown of FL theory must be more subtle than is found in any of the traditional perturbation theory approaches. The present calculation cannot address the issue raised by Anderson [4,10] as to the validity of perturbation theory in 2D except to say that there is no indication of this from perturbation theory itself.

Apart from the question of stability of the Fermi liquid our results for the corrections to Fermi liquid theory are most surprising. In 3D the specific heat is known to have a $T^3\ln T$ correction, i.e., $C_V \simeq \gamma T + \Gamma_{3\mathrm{D}} T^3 \ln T$. In the 1D system the breakdown of the FL can be seen already in the second order perturbation theory where the specific heat correction is given by $\delta C_V \simeq \Gamma_{1\mathrm{D}} T \ln T$. (This is a clear signal that perturbation theory does not work since this is more important than the linear term.) In the 2D case we find that $C_V \simeq \gamma T + \Gamma_{2\mathrm{D}} T^2 + \cdots$. One might have expected this to have a $T^2 \ln T$ correction by studying the 3D and 1D behavior, this in fact is not the case.

In order to determine the leading corrections to a 2D FL due to long wavelength interactions we consider a system of fermions which interact via a two-body potential as in Eq. (1):

$$H = \sum_{\mathbf{p},\sigma} \frac{|\mathbf{p}|^2}{2m} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma} + \sum_{\mathbf{p},\mathbf{q},\alpha,\beta,\gamma,\delta} V_{\alpha,\beta,\gamma,\delta}(\mathbf{q}) c^\dagger_{\mathbf{p},\alpha} c_{\mathbf{p}',\beta} c^\dagger_{\mathbf{p}'-\mathbf{q},\gamma} c_{\mathbf{p}+\mathbf{q},\delta} . \quad (1)$$

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Expanding in the particle-hole channel the effect of the interaction may be considered as coming from two independent channels, the symmetric (s) and the antisymmetric (a) channels corresponding to no spin exchanged and to spin 1 exchanged. Using the paramagnon model first introduced by Doniach and Engelsberg [11], for the $\mathbf{q}$ dependence of the interaction, the value of the interaction in the symmetric channel, V_s, is $-I/2$ and in the antisymmetric channel, V_a, is $2I$, where $I>0$ [12]. I is the strength of the interaction and multiplied by the density of states of one spin is the paramagnon parameter, $\bar{I}$. The interaction is cut off at $|\mathbf{q}|=q_c$. Our results for the paramagnon model may be easily generalized to the Landau theory of FL's by replacing the paramagnon parameter by Landau parameters since we concentrate on long wavelength effects in a particle-hole expansion. We discuss this in more detail towards the end of the paper. So our results are much more general than the paramagnon model and hold for any interaction for which perturbation theory is valid. By using the paramagnon model we are able to investigate the long wavelength properties of FL's without having to worry about details of the momentum dependence of the potential.

In order to compare the properties of FL's in 2D and 3D we first calculate the single-particle self-energy to second order in perturbation theory at zero temperature. We consider the real and imaginary parts of $\Sigma(\mathbf{p},\epsilon)$ separately in order to make contact with the known results for $\mathrm{Im}\Sigma(\mathbf{p},\xi_\mathbf{p})$. We calculate $\mathrm{Re}\Sigma(\mathbf{p},\epsilon)$ using $\mathrm{Im}\Sigma(\mathbf{p},\epsilon)$ via the Kramers-Kronig relation which necessitates our calculating $\mathrm{Im}\Sigma(\mathbf{p},\epsilon)$ for all ϵ not just on shell. In order to discover the functional form of $\Sigma(\mathbf{p},\epsilon)$ for a FL it is sufficient to go to second order in the interaction. For a 3D FL Blaizot and Friman [13] found

$$\mathrm{Im}\Sigma(\mathbf{p},\epsilon)=\frac{\pi\bar{I}^2}{8v_F^2p_F^2}\mathrm{sgn}(\epsilon)\times([q_cv_F-|\xi_p|]\epsilon^2+\tfrac{1}{3}|\epsilon|^3+\cdots), \quad (2)$$

where $\xi_p=(p-p_F)v_F$, p_F is the Fermi momentum, and $v_F=p_F/m$ is the Fermi velocity. The real part of $\Sigma(\mathbf{p},\epsilon)$ is determined from the Kramers-Kronig relation,

$$\mathrm{Re}\Sigma(\mathbf{p},\epsilon)=\frac{1}{\pi}P\int_{-\infty}^{\infty}d\zeta\frac{\mathrm{Im}\Sigma(\mathbf{p},\zeta)}{\epsilon-\zeta} \quad (3)$$

and is given by

$$\mathrm{Re}\Sigma(\mathbf{p},\xi_p)=A_{3\mathrm{D}}\xi_p+B_{3\mathrm{D}}\xi_p^3\ln|\xi_p|+\cdots. \quad (4)$$

The $\xi_p^3\ln|\xi_p|$ term comes from the $|\epsilon|^3$ term in $\mathrm{Im}\Sigma(\mathbf{p},\epsilon)$ and so it is determined by the long-wavelength scattering. In fact it has been shown by Moriya [14] that no $\xi_p^3\ln|\xi_p|$ terms come from finite $\mathbf{q}$ scattering. For a 2D FL we find

$$\mathrm{Im}\Sigma(\mathbf{p},\epsilon)=\mathrm{sgn}(\epsilon)\frac{4\bar{I}^2}{2\pi N(0)v_F^2}\{[\xi_p^2+2\xi_p(\epsilon-\xi_p)]\ln(\max[\xi_p,|\epsilon|])\Theta(q_cv_F-|\epsilon|)+(\epsilon-\xi_p)^2\ln(|\epsilon-\xi_p|)\Theta(q_cv_F-|\epsilon-\xi_p|)+\cdots\}, \quad (5)$$

for $\xi_p>0$. This reduces to the well-known results of Hodges, Smith, and Wilkins [15] and Bloom [16] for $\mathrm{Im}\Sigma(\mathbf{p},\epsilon=\xi_p)$. Using the Kramers-Kronig relation again, $\mathrm{Re}\Sigma(\mathbf{p},\xi_p)$ is

$$\mathrm{Re}\Sigma(\mathbf{p},\xi_p)=A_{2\mathrm{D}}\xi_p+B_{2\mathrm{D}}\mathrm{sgn}(\xi_p)\xi_p^2+\cdots. \quad (6)$$

Only the first term in $\mathrm{Im}\Sigma(\mathbf{p},\epsilon)$ contributes to $\mathrm{Re}\Sigma(\mathbf{p},\xi_p)$. The linear terms in $\mathrm{Re}\Sigma(\mathbf{p},\xi_p)$ for 2D and 3D are effective mass enhancements, which come from all $\mathbf{q}$'s and depend on the $\mathbf{q}$ dependence of the interaction. Here we are concerned with the corrections to the effective mass enhancement terms. As in the case of $B_{3\mathrm{D}}$, $B_{2\mathrm{D}}$ is determined by long wavelength scattering. Comparing the corrections to the effective mass terms in $\mathrm{Re}\Sigma(\mathbf{p},\xi_p)$ in 2D and 3D, one sees that in 2D the correction is nonanalytic and comes from the leading ϵ dependence of $\mathrm{Im}\Sigma(p,\epsilon)$ whereas in 3D the correction is analytic but comes from nonanalytic terms in $\mathrm{Im}\Sigma(p,\epsilon)$. This difference between the leading corrections to the linear ξ_p dependence in 2D and 3D is due solely to the different phase space. This may be seen by calculating the contribution to the spectrum in 2D using the equation

$$\Delta\epsilon_p=\sum_{|\mathbf{q}|<q_c}[1-2f_{\mathbf{p}+\mathbf{q}}](\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2V^{(2)}, \quad (7)$$

where $V^{(2)}$ is the coefficient of the $(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2$ term in the effective quasiparticle interaction. In 3D this gives the $\xi_p^3\ln|\xi_p|$ dependence [17]. In contrast to 1D, where the analogous calculation already shows that FL theory has broken down, there is no indication of a breakdown of FL theory in 2D to this order in perturbation theory. We now consider the thermodynamics of a 2D FL and compare the results with 3D.

Using the RPA approximation the change in the thermodynamical potential due to interactions in Eq. (1) is given by

$$\Delta\Omega=k_BT\sum_{\mathbf{q},\omega_n}(\tfrac{3}{2}\{\ln[1-I\chi(\mathbf{q},\omega_n)]+I\chi(\mathbf{q},\omega_n)\}+\tfrac{1}{2}\{\ln[1+I\chi(\mathbf{q},\omega_n)]-I\chi(\mathbf{q},\omega_n)\}), \quad (8)$$

where

$$\chi(\mathbf{q},\omega)=2\sum_{\mathbf{p}}\frac{f_{\mathbf{p}+\mathbf{q}}-f_\mathbf{p}}{\omega-(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_\mathbf{p})}, \quad (9)$$

$f_\mathbf{p}$ is the Fermi-Dirac distribution function, and $\omega_n=2\pi\times(n+1)T$ are Matsubara frequencies. Although a calculation of $\Delta\Omega_{\mathrm{qp}}$ to second order in the interactions gives the functional form of the temperature dependence, by calculating $\Delta\Omega_{\mathrm{qp}}$ in the RPA approximation we are able to point to a difference between the coefficients of the leading corrections to the specific heat of 2D and 3D FL's which we point out below. When analytically continued to the real ω axis $\Delta\Omega$ can be easily broken up into a quasiparticle contribution, $\Delta\Omega_{\rm qp}$, and a contribution from collective modes, $\Delta\Omega_{\rm coll\ modes}$. First we consider $\Delta\Omega_{\rm qp}$ which is given by [12]

$$\Delta\Omega_{\rm qp}=\sum_{|\mathbf{q}|<q_c}\int_0^\infty \frac{d\omega}{\pi} n_B(\omega)[F(\mathbf{q},\omega)+I\chi''(\mathbf{q},\omega)]\,, \tag{10}$$

$$F(\mathbf{q},\omega)=\frac{3}{2}\tan^{-1}\left[\frac{-I\chi''(\mathbf{q},\omega)}{1-I\chi'(\mathbf{q},\omega)}\right]+\frac{1}{2}\tan^{-1}\left[\frac{I\chi''(\mathbf{q},\omega)}{1+I\chi'(\mathbf{q},\omega)}\right], \tag{11}$$

$$\chi(\mathbf{q},\omega)=\chi'(\mathbf{q},\omega)+i\chi''(\mathbf{q},\omega)\,, \tag{12}$$

and $n_B(\omega)$ is the Bose distribution function. From this the change in the entropy is

$$\Delta S_{\rm qp}=-\left[\frac{\partial\Delta\Omega_{\rm qp}}{\partial T}\right]_\mu=\sum_{|\mathbf{q}|<q_c}\int_0^\infty\frac{d\omega}{\pi}\left(\frac{\partial n_B(\omega)}{\partial T}\bigg|_\mu F(\mathbf{q},\omega)+n_B(\omega)\frac{\partial F(\mathbf{q},\omega)}{\partial T}\bigg|_\mu+I\frac{\partial[n_B(\omega)\chi''(\mathbf{q},\omega)]}{\partial T}\bigg|_\mu\right). \tag{13}$$

The two terms on the right of Eq. (13) involve the temperature dependence of $\chi(\mathbf{q},\omega)$ which is weak when μ is kept constant.

Calculating $\Delta S_{\rm qp}$ for 2D one finds

$$\Delta S_{\rm qp}=\gamma'_{2D}T+\Gamma_{2D}T^2+O(T^3)\,, \tag{14}$$

$$\gamma'_{2D}=\frac{\pi}{6T_F}(A_s+A_a)\frac{q_c}{p_F}\,,$$
$$\Gamma_{2D}=\frac{\pi n}{4T_F^2}\sum_\lambda \nu_\lambda\left[A_\lambda+\int_0^1 du\, f_\lambda(u)\right], \tag{15}$$

where

$$f_\lambda(u)=\frac{\{A_\lambda u-\tan^{-1}[A_\lambda u/(1-u^2)]\}}{u^3}\,, \tag{16}$$

where $\lambda=s$ or a, $\nu_s=1$, $\nu_a=3$, and

$$A_s=\frac{\bar{I}}{1+\bar{I}},\quad A_a=\frac{-\bar{I}}{1-\bar{I}}\,. \tag{17}$$

A_s and A_a are the scattering amplitudes in the symmetric (density) and antisymmetric (spin) channels, n is the density of particles, and $T_F=v_Fp_F/2$. The T^2 term in $\Delta S_{\rm qp}$ comes from the nonanalytic term in $\mathrm{Re}\Sigma(\mathbf{p},\xi_p)$ as may be seen from the following argument [18]. Consider the entropy of a FL whose spectrum is given by

$$\epsilon_p=\xi_p+\Delta\epsilon_p\,, \tag{18}$$

where $\Delta\epsilon_p$ arises from interactions. Substituting this spectrum into the expression for the entropy of a noninteracting Fermi liquid and expanding to linear order in $\Delta\epsilon_p$ one finds

$$\Delta S=\sum_{\mathbf{p}}\frac{\xi_p}{T}\frac{\Delta\epsilon_p}{T}\frac{\partial f(\epsilon)}{\partial\epsilon}\bigg|_{\xi_p}\,. \tag{19}$$

Assuming that $\Delta\epsilon_p$ can be expanded in a power series in ξ_p one finds

$$\Delta S=N(0)k_B\int_{-\infty}^\infty\frac{\xi d\xi}{T^2}\frac{1}{4\cosh^2(\xi/2T)}\sum_n a_n\xi^n\,. \tag{20}$$

One sees that only odd n terms contribute to ΔS and that they lead to series of odd powers of T. The presence of the T^2 in $\Delta S_{\rm qp}$ clearly arises from the nonanalytic nature of the correction to the spectrum in Eq. (6). The terms of $O(T^3)$ and higher are a sum of odd powers of temperature.

Carrying out the calculations in 3D one finds that

$$\Delta S_{\rm qp}=\gamma'_{3D}T+\Gamma_{3D}T^3\ln T+O(T^3)\,. \tag{21}$$

Another difference between 2D and 3D is that Γ_{2D} depends on the scattering amplitudes to all orders whereas Γ_{3D} involves only the second and third powers of the scattering amplitudes.

The collective mode contribution in 2D is

$$\Delta S_{\rm coll\ mode}=\Gamma''T^2+O(T^4)\,, \tag{22}$$

where

$$\Gamma''=\frac{1}{2\pi}\left(\frac{1}{c}\right)^2 \tag{23}$$

and c is the velocity of the collective mode given by

$$c=v_F\frac{1+\bar{I}}{\sqrt{1+2\bar{I}}}\,. \tag{24}$$

The collective mode spectrum does not contain any log dependence on $|\mathbf{q}|$ and so there is no $\ln T$ contribution to $\Delta S_{\rm coll\,mode}$. In 3D $\Delta S_{\rm coll\,mode}\sim T^3$ and does not contribute to the $T^3\ln T$ corrections except to change the cutoff of the logarithmic temperature dependence.

Collecting $\Delta S_{\rm qp}$ and $\Delta S_{\rm coll\,mode}$ together one sees that ΔS is a power series in temperature in 2D. In particular, there are no $\ln T$ terms in ΔS which implies that, at least to leading order, quasiparticle damping effects do not contribute to thermodynamic properties in this approximation where the propagators are unrenormalized. The effect of finite quasiparticle lifetimes on the entropy may be estimated with Eq. (23) of Ref. [19] and is given by

$$\Delta S_{\text{damp}} = \sum_{\mathbf{p}} \int_{-\infty}^{\infty} d\epsilon \frac{\partial f(\epsilon)}{\partial T} G(\lambda(\mathbf{p},\epsilon)) , \tag{25}$$

where

$$G(\lambda(\mathbf{p},\epsilon)) = \frac{\lambda(\mathbf{p},\epsilon)}{1+\lambda(\mathbf{p},\epsilon)^2} - \tan^{-1}[\lambda(\mathbf{p},\epsilon)] \tag{26}$$

and

$$\lambda(\mathbf{p},\epsilon) = -\frac{\mathrm{Im}\Sigma(\mathbf{p},\epsilon)}{\mathrm{Re}\Sigma(\mathbf{p},\epsilon)} . \tag{27}$$

The functional form of the integrand in Eq. (25) is very complicated and analytic evaluation is intractable. So we content ourselves with an estimate. $G(\lambda(\mathbf{p},\epsilon))$ is a smooth function of $\lambda(\mathbf{p},\epsilon)$ which goes as $\lambda(\mathbf{p},\epsilon)^3$ for small $\lambda(\mathbf{p},\epsilon)$ and is a constant for large $\lambda(\mathbf{p},\epsilon)$. In order to get an estimate of ΔS_{damp} we assume that $G \sim \lambda(\mathbf{p},\epsilon)^3$ for all values of ϵ. Since there is no contribution to the integral for large values of $|\lambda(\mathbf{p},\epsilon)|$, this is clearly an overestimate. With these approximations one finds $\Delta S_{\text{damp}} \sim T^7 \ln^3 T$ which is higher order in temperature than the T^2 corrections found above. Lifetime effects lead to higher order effects in the thermodynamics than T^2 so that they are much less important in thermodynamics than a calculation of $\Sigma(\mathbf{p},\epsilon)$ would suggest.

Since we have used a particle-hole expansion in the symmetric and antisymmetric channels, our results may be easily extended to Landau's Fermi liquid theory by considering Eq. (1) to describe quasiparticles with an effective mass interacting via an effective interaction $f(\mathbf{p},\mathbf{p}')$ which can be decomposed into two channels, $f_s(\mathbf{p},\mathbf{p}')$ and $f_a(\mathbf{p},\mathbf{p}')$. This effective interaction is a long wavelength limit of the particle-hole irreducible four-point vertex and so describes long wavelength properties. This leads to somewhat more complicated expressions when V_s and V_a are substituted for by $f_s(\mathbf{p},\mathbf{p}')$ and $f_a(\mathbf{p},\mathbf{p}')$. The Landau functions are functions of the variable $s = \omega/qv_F$ and may be expressed as coefficients in a series of Legendre polynomials in which s is the argument. These coefficients are the Landau parameters.

The present calculation indicates that a 2D FL is very similar to the 3D case and that any breakdown of the FL in 2D has to arise from effects which are more subtle than those which give the leading corrections to FL theory in 3D. The logarithmic dependence in $\mathrm{Im}\Sigma(\mathbf{p},\epsilon)$ allows us to keep track of the contribution to the thermodynamic properties from lifetime effects. We find that in spite of the $\xi_p^2 \ln \xi_p$ dependence of the relaxation time in 2D their contributions are higher order in T than the contributions to the quasiparticle spectrum.

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