Transverse Phase-Space Dynamics of Mismatched Charged-Particle Beams

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The transverse dynamics of a nonrelativistic, mismatched charged-particle beam propagating through a continuous, linear focusing channel is calculated using the Fokker-Planck equation to represent the evolution of a coarse-grained distribution function in the phase space of a single beam particle. The relaxation rate and diffusion coefficient are determined from a simple model of turbulence resulting from charge redistribution. The solution for the distribution function enables calculation of all transverse beam properties as a function of time, including the halo.

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This paper concerns the dynamics of transverse emittance growth and halo formation in nonrelativistic space-charge-dominated beams propagating through continuous, linear focusing channels. Mismatches in the density profile and beam size contribute to emittance growth [1, 2]. A mismatched beam carries excess energy which can be thermalized if nonlinear forces, instabilities, and/or collisions are present. Thermalization generates emittance growth, the magnitude of which can be calculated from the excess energy [2]. It also ejects beam particles into high-amplitude orbits, thereby generating a halo. These effects are detrimental to many advanced accelerator applications. For example, applications requiring ion beams with high current and low emittance include linear accelerators for heavy-ion-ignited inertial fusion. Applications requiring minimal transverse halo include linear accelerators envisioned for long-term, continuous-wave operation, in which impingement of beam particles on the accelerating structures may generate radioactivation, thereby inhibiting routine maintenance. Conventional design philosophies relying exclusively on controlling root-mean-square (rms) properties of the beam are insufficient for these accelerators.

Charge redistribution begins when the beam enters the focusing channel. In a zero-temperature beam, particle trajectories initially do not cross, and laminar flow is present. However, if the initial density gradually falls to zero at the beam edge, then laminar flow ceases very quickly, at about one-quarter of a plasma period after injection, at which time particle trajectories originating in that part of the beam with the lowest initial density will cross [3]. Laminar flow terminates with nonlinearities in the form of discontinuous shocklike behavior associated with wave breaking in phase space and the onset of irreversible dynamics [3, 4]. Charge redistribution in a warm beam terminates similarly [3].

While the charge-redistribution phase permits simple analysis, subsequent evolution is more complicated. Understanding this evolution is essential because it dominates almost the entire transport of real beams. Prior investigators approached this problem using both direct experimentation [5, 6] and computational N-body simulation [1, 7, 8]. The purpose of this paper is to identify the essential underlying physics and integrate it into a single formalism accounting for the salient dynamical features discovered in prior investigations.

We select an arbitrary beam cross section and discuss its evolution from the perspective of a comoving frame. During charge redistribution, the beam evolves toward a density profile which is nearly uniform, particularly if space charge is strong. If this nearly uniform beam is mismatched in size to the transport channel, it carries free energy available for thermalization. The shocklike behavior and wave breaking in phase space ending this phase are nonlinear phenomena which may strongly excite random collective degrees of freedom, thereby establishing turbulence. Coupling between beam particles and the turbulence provides a mechanism for converting free energy, now contained in the turbulent fluctuations, into heat. In strong turbulence the heating will occur very rapidly, on a time scale of the order of a plasma period [9]. Because it occurs at the expense of the energy contained in the turbulent fluctuations, the turbulence also weakens on the same time scale.

Beam particles slow down by interacting with turbulent fluctuations of the net electric field. This anomalous resistivity also occurs on a time scale of the order of a plasma period in strong turbulence. In three-dimensional beams, the associated average collision frequency is then $\sim g^{-1} \equiv n\lambda_D^3$ times larger than in a quiescent plasma, where n is the particle density and λ_D is the Debye length [9]. In space-charge-dominated beams g^{-1} is large and these interactions are important. In weak turbulence the average collision frequency is $\sim g^{1/2}$ times smaller than in strong turbulence.

Interactions with turbulent fluctuations are considered to impart white noise on particle trajectories, thereby establishing a Markoff process resulting in Brownian motion. The interactions create dynamical friction and diffusion, causing relaxation to occur on a time scale which evolves from short to long as the turbulence dissipates. Heating and relaxation of the beam eject a fraction of the particles into large-amplitude orbits causing the emittance to grow and a transverse halo to form. Because the relaxation time can be very short, these processes can occur during beam transport. Therefore, even though from the perspective of discrete-particle interactions the beam may be regarded as collisionless during transport, collective interactions established by turbulence cause effects similar to collisions in that they tend to smear the particles in phase space, destroying ordered motion and establishing a thermal velocity distribution. Phase space becomes covered in a coarse-grained manner [10], and the dynamics may therefore be represented with a coarse-grained distribution function of particles in the phase space of a single particle.

These considerations motivate using the Fokker-Planck equation to govern the coarse-grained distribution function. We let $W(\mathbf{x},\mathbf{u},t;\mathbf{x}_0,\mathbf{u}_0)$ denote the coarse-grained probability of finding a particle with position $\mathbf{x}$ and velocity $\mathbf{u}$ in the range $(\mathbf{x},\mathbf{x}+d\mathbf{x})$ and $(\mathbf{u},\mathbf{u}+d\mathbf{u})$, respectively, at time t given it started at $(\mathbf{x}_0,\mathbf{u}_0)$ at $t=0$. The Fokker-Planck equation is

$$\partial_t W+\mathbf{u}\cdot\boldsymbol{\nabla}_x W+\mathbf{K}\cdot\boldsymbol{\nabla}_u W=\beta\boldsymbol{\nabla}_u\cdot(W\mathbf{u})+D\nabla_u^2 W, \quad (1)$$

where $\mathbf{K}$ is the net force per particle mass m, β is the relaxation rate, and D is the diffusion coefficient. Both the probability and the net force are regarded to be smoothed out. The effects of turbulent fluctuations about the mean electric field, dynamical friction and diffusion, are represented by the Fokker-Planck collision operator on the right-hand side of Eq. (1). Though the beam is generally out of equilibrium, we define and use a "temperature" $T\equiv mD/\beta k$, where k is Boltzmann's constant. In equilibrium this relationship gives the true thermodynamic temperature. The net force in the comoving frame is the superposition of the focusing force and the mean space-charge force found from the potentials Φ_f and Φ_s, respectively, and thus

$$\mathbf{K}=-qm^{-1}\boldsymbol{\nabla}_x(\Phi_f+\Phi_s)\ , \quad (2)$$

where q is the particle charge. According to Poisson's equation, the mean space-charge potential Φ_s is determined from the smoothed-out density, which in turn is determined from the coarse-grained probability:

$$\nabla_x^2\Phi_s(\mathbf{x},t)=-\frac{\mathcal{N}q}{\varepsilon_0}\iiint d\mathbf{u}\,d\mathbf{u}_0\,d\mathbf{x}_0\,W(\mathbf{x},\mathbf{u},t;\mathbf{x}_0,\mathbf{u}_0)\,W(\mathbf{x}_0,\mathbf{u}_0)\ , \quad (3)$$

where $\mathcal{N}$ is a normalization parameter related to the particle density in the beam, ε_0 is the permittivity of free space, and $W(\mathbf{x}_0,\mathbf{u}_0)$ is the single-particle distribution function at $t=0$.

The relaxation rate and diffusion coefficient with its associated temperature will generally depend on both position and time. However, if turbulence is initially strong enough to induce rapid heating and relaxation, and if most beam particles interact with the strong fluctuations as they orbit, then to a reasonable approximation the beam may be regarded to have a uniform, decreasing relaxation rate and a uniform, increasing temperature which saturates as turbulence weakens. Letting β_s denote the relaxation rate in strong turbulence, we adopt an exponential model of turbulent heating:

$$T(t)=T_\infty+(T_0-T_\infty)e^{-\beta_s t}\ . \quad (4)$$

Starting from temperature T_0, the beam strives to reach a Maxwell-Boltzmann distribution with temperature T_∞.

To obtain a self-consistent solution, one must solve Eqs. (1)–(3) simultaneously using Eq. (4) to represent turbulent heating. For mathematical simplicity we treat a one-dimensional (1D) rms-mismatched sheet beam centered on the focusing-channel axis which provides a focusing force $-m\omega^2 x$. Because the charge-redistribution phase lasts for a very short time, we ignore its detailed dynamics and regard the turbulence resulting from charge redistribution to be present at $t=0$, the time of injection of the beam into the focusing channel. Our treatment may be straightforwardly generalized to higher dimensions, and the qualitative features of the results in the presence of turbulence should be insensitive to the dimensionality. However, the onset of turbulence may depend strongly on the dimensionality, an effect which has been observed in some numerical simulations [11]. This circumstance may be analogous to the sensitivity of phase transitions to dimensionality.

Assuming $\beta\ll\beta_s$ to be a constant relaxation rate in weak residual turbulence, and using a harmonic-oscillator model of the particle orbits, we can solve Eq. (1) in closed form with standard methods [12]. We take $-m\omega_n^2 x$ to be the net restoring force, where ω_n is the particles' orbital frequency. This replaces Poisson's equation, and thus self-consistency is sacrificed. However, the approach is instructive because it provides simple results exhibiting many of the prominent features of self-consistent solutions discussed below. Moreover, the resulting distribution function can be used to calculate any moment of x and u in terms of elementary functions.

Following Ref. [12], we express the solution of the Fokker-Planck equation in terms of the variables $\xi=(\mu_1 x-u)e^{-\mu_2 t}$, $\eta=(\mu_2 x-u)e^{-\mu_1 t}$ and their initial values ξ_0, η_0, in which $\mu_1=-(\beta-\beta_1)/2$, $\mu_2=-(\beta+\beta_1)/2$, and $\beta_1=(\beta^2-4\omega_n^2)^{1/2}$. The solution which tends to $\delta(\xi-\xi_0)\,\delta(\eta-\eta_0)$ as $t\to 0$ is

$$W=\frac{m}{2\pi kT_0}\frac{e^{\beta t}}{\sqrt{\Delta}}\exp\left\{-\frac{m}{2kT_0\Delta}\left[a(\xi-\xi_0)^2+2h(\xi-\xi_0)(\eta-\eta_0)+b(\eta-\eta_0)^2\right]\right\}\ , \quad (5)$$

where a, b, and h are dimensionless functions of time found by integrating $2\beta T/T_0$ multiplied by $e^{-2\mu_1 t}$, $e^{-2\mu_2 t}$, or $-e^{-(\mu_1+\mu_2)t}$, respectively, over the interval $[0,t]$, and $\Delta = ab - h^2$. These functions are easily evaluated from the model of turbulent heating, Eq. (4).

The coarse-grained probability function given in Eq. (5) can be used to calculate the density function governing the average spatial distribution of the particles and any other smoothed-out moment of x, u in terms of elementary functions. Two moments of special interest are the rms beam size $X = \langle x^2 \rangle^{1/2}$ and rms emittance $\epsilon = (\langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2)^{1/2}$. If the beam enters the accelerator with a Maxwellian velocity distribution, these quantities, normalized to their values at $t = 0$, are respectively found to be

$$\frac{X}{X_0} = \frac{e^{-\beta t/2}}{\beta_1}\left\{e^{\beta_1 t}\left[\Omega^2(a+1)+\mu_2^2\right] + e^{-\beta_1 t}\left[\Omega^2(b+1)+\mu_1^2\right] + 2\left[\Omega^2(h-1)-\omega_n^2\right]\right\}^{1/2} , \tag{6}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{e^{-\beta t}}{\beta_1}\left[\Omega^2\Delta + a(\Omega^2+\mu_1^2) + b(\Omega^2+\mu_2^2) + 2h(\Omega^2+\omega_n^2) + \beta_1^2\right]^{1/2} , \tag{7}$$

where $\Omega = \omega_n T_0 \epsilon_\infty / T_\infty \epsilon_0$, $T_\infty/T_0 = (X_0\epsilon_\infty/X_\infty\epsilon_0)^2$, and X_0, ϵ_0; X_∞, ϵ_∞ refer to $t=0$ and $t \to \infty$, respectively. The degree of mismatch determines the ratios X_∞/X_0 and $\epsilon_\infty/\epsilon_0$ [2]. Example plots of X and ϵ in the limit of instantaneous heating appear in Fig. 1(a). In this model, the rise time of the emittance corresponds to the relaxation rate β of residual, weak turbulence, even with instantaneous heating from initially strong turbulence.

We now solve Eqs. (1)–(3) self-consistently by decomposing the single-particle distribution function into complete sets of orthonormal polynomials [13]. For the 1D sheet beam a natural choice is the set of orthonormal Hermite functions $\phi_m(\sqrt{A}x)$ and $\psi_n(\sqrt{\alpha}u)$, where $A = m\omega_0^2/2kT$, ω_0 is a reference frequency, and $\alpha = m/2kT$. This allows us to express the distribution function in a form which is an eigenfunction of the Fokker-Planck collision operator:

$$W(x,u,t) = \phi_0\psi_0 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n^m \phi_m \psi_n , \tag{8}$$

in which c_n^m are time-dependent coefficients. By symmetry, the coefficients c_n^m for which $m+n$ is odd are zero for a centered beam. The leading term of the distribution function is Gaussian in both velocity space and configuration space. This is advantageous because laboratory beams injected into focusing channels may have properties close to these, and so the initial conditions are easy to incorporate. Moreover, this is the equilibrium distribution for a relaxed beam with zero space-charge force in a linear focusing channel, so Eq. (8) includes a good foundation on which to build the effects of nonlinear space-charge forces.

The coarse-grained particle density, beam size, and emittance are respectively given by

$$n = \mathcal{N}\phi_0 \sum_{p=0}^{\infty} c_0^{2p}\phi_{2p} , \tag{9}$$

$$X/X_0 = [(T/T_0)(1+\sqrt{2}\,c_0^2)]^{1/2} , \tag{10}$$

$$\epsilon/\epsilon_0 = (T/T_0)\left[(1+\sqrt{2}\,c_0^2)(1+\sqrt{2}\,c_2^0) - (c_1^1)^2\right]^{1/2} . \tag{11}$$

Integrating n over all values of x must always yield $\mathcal{N}$; consequently, we must have $c_0^0 = 1$ for all t. The turbulence is expected to pass rapidly from strong to weak, so we use Eq. (4) for turbulent heating and, as before, we let $\beta \ll \beta_s$ be a constant in Eq. (1) to represent persistent weak turbulence.

Upon substituting for $W(x,u,t)$ in Eqs. (1)–(3) and using the orthonormality and recurrence relations of ϕ_j and ψ_k, we obtain the following infinite system of coupled differential equations for the coefficients c_n^m which is equivalent to the coupled Fokker-Planck and Poisson equations:

$$\begin{aligned}\dot{c}_n^m = {} & -(\dot{T}/2T)\left[\sqrt{m(m-1)}\,c_n^{m-2} + \sqrt{n(n-1)}\,c_{n-2}^m + (m+n)\,c_n^m\right] - n(\beta/\omega_0)\,c_n^m \\ & + \sqrt{m}\left(\sqrt{n}\,c_{n-1}^{m-1} + \sqrt{n+1}\,c_{n+1}^{m+1}\right) - (\omega/\omega_0)^2\sqrt{n}\left(\sqrt{m}\,c_{n-1}^{m-1} + \sqrt{m+1}\,c_{n-1}^{m+1}\right) \\ & + (\omega_p/\omega_0)^2(T_0/T)^{1/2}\sqrt{2n}\sum_{q=0}^{\infty} c_{n-1}^q \kappa_m^q ,\end{aligned} \tag{12}$$

where the dot denotes differentiation with respect to $\zeta = \omega_0 t$, $\omega_p = (\mathcal{N}q^2\omega_0/2\varepsilon_0\sqrt{\pi m k T_0})^{1/2}$ is the plasma frequency, and

$$\kappa_p^q = -\sum_{n=0}^{\infty} c_0^{2n}\frac{\Gamma[n+(q-p)/2]\,\Gamma[1-n+(q+p)/2]\,\Gamma[n-(q-p)/2]}{\pi\sqrt{2\pi p!q!(2n)!}} . \tag{13}$$

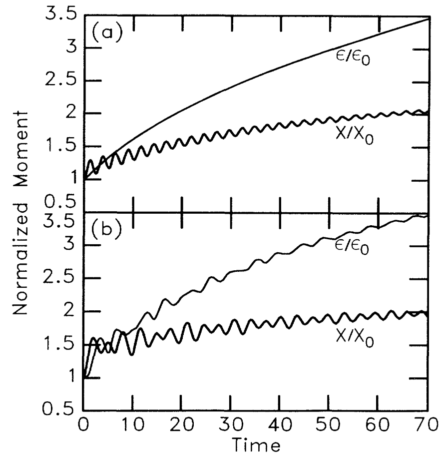

FIG. 1. rms beam size and emittance, normalized to initial values, (a) from harmonic-oscillator model of particle orbits with $(\omega_n/\omega_0)^2 = 1.5$, $\beta = 0.015$, $X_\infty/X_0 = 2.4$, $\epsilon_\infty/\epsilon_0 = 4.8$ in the limit $\beta_s \to \infty$; (b) calculated self-consistently with $(\omega/\omega_0)^2 = 1.5$, $(\omega_p/\omega_0)^2 = 1.4$, $\beta_s/\omega_0 = 1.0$, $\beta/\omega_0 = 0.015$, and $T_\infty/T_0 = 4$.

Upon truncating at $m = M$, $n = N$, the system can be integrated numerically. We set $M = 4$, $N = 3$ to allow for two time-dependent coefficients c_0^2, c_0^4 in Eq. (9) for the density. This is sufficient to illustrate the physics; however, numerical accuracy can be improved as desired by continuously increasing M, N until essentially no change is seen in the results. The truncated system is solved for a beam entering the focusing channel with a Maxwellian velocity distribution and a Gaussian density profile of standard deviations $(kT_0/m)^{1/2}$ and $(kT_0/m\omega_0^2)^{1/2}$, respectively. The initial conditions are $c_n^m = \delta_{mn}$.

Example plots of X and ϵ in a self-consistent calculation appear in Fig. 1(b). The parameters were chosen to be consistent with those of Fig. 1(a) representing the analytic solution. Though their details differ, the qualitative features of the two solutions are similar. In particular the relaxation rate β determines the rise time of the emittance. The self-consistent solution was verified to give the same results for the rms quantities as the analytic solution in the limit of zero space charge ($\omega_p = 0$, $\omega_n = \omega$); thus, differences between the curves in Fig. 1 are attributed to the space-charge force.

As the beam relaxes, more and more particles are ejected into high-amplitude orbits. The spreading density profile may generate excessive radioactivation if the transport elements have small bore-hole apertures. Since growth in beam size and emittance correlates to degree of mismatch, the cure involves both reducing sources of mismatch and increasing the bore-hole sizes.

We based our calculations on a simple model of turbulence which was assumed to be strong initially, causing rapid heating of the beam, and to dissipate quickly to persistent weak turbulence. The relaxation rate was taken to be a constant associated with weak turbulence and was regarded as a free parameter. Since the spectrum of electric-field fluctuations determines the relaxation rate and diffusion coefficient, it would be interesting to use N-body simulations to relate the onset of turbulence and the time dependence of the fluctuation spectrum to accelerator design parameters. These studies may lead to more accurate Fokker-Planck coefficients for use in the semianalytic formulation.

Though we considered a sheet beam in a constant focusing force, the formalism can be adapted to higher-dimensional beams and nonstationary focusing forces. It can also be used to explain and describe the recurrence and gradual dissipation of fine structure present in the beam at injection, a dynamical phenomenon seen in transport experiments and numerical simulations [5, 6].

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