Self-Organized Fractal River Networks

A. Rinaldo,(1) I. Rodriguez-Iturbe,(2),(3) R. Rigon,(1) E. Ijjasz-Vasquez,(3) and R. L. Bras(3)
(1)*Istituto di Idraulica "G. Poleni," Università di Padova, via Loredan 20, I-35131 Padova, Italy*
(2)*Instituto Internacional de Estudios Avanzados, P.O. Box 17606, Caracas, Venezuela*
(3)*Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
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Optimal channel networks (OCN's) obtained by minimizing the local and global rates of energy expenditure evolve automatically from arbitrary initial conditions to network configurations exhibiting fractal and multifractal statistics indistinguishable from those observed in nature. It is suggested that OCN's are spatial models of self-organized criticality and that natural fractal structures like river networks may arise as a joint consequence of optimality and randomness.

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The mechanisms of formation of fractal patterns have been studied in various fields of science [1]. Using models of optimal channel networks (OCN's) [2] displaying spatial self-organized criticality [3], this Letter suggests that the formation process of fractal river networks achieves a steady state upon the attainment of local optimal rules equivalent to a critical erosion parameter and from global rules of minimum energy dissipation equivalent to stable equilibrium. The studied system always evolves into a stable critical state regardless of the initial conditions.

River patterns show consistently fractal [4,5] and multifractal [2,5,6] characteristics through experimental analysis of digital elevation maps (DEM's) [5]. Power laws, which are the signature of fractal behavior, have been experimentally observed over a huge range of scales in probability distributions describing river basin morphology. Notably, the cumulative total drainage area A contributing to any link follows a power law as $P[A > a] \propto a^{-0.43 \pm 0.03}$ [5]; the stream lengths L follow a power-law distribution with exponent close to 2 [4]; the mean of the local slope ∇z of the links of a drainage network scales as a function of cumulative area A according to $E[\nabla z(A)] \propto A^{-0.5}$ [2,4]; Horton's power laws of bifurcation and length [7] are verified [4] suggesting analytic derivations of fractal dimensions [4,6]. [A link is defined as a contiguous collection of sites which share the same flow, to which an area is assigned [7]; drainage directions follow the local slope ∇z, z being the landscape elevation, and define a network as a forest of rooted subtrees [1,2,4]; the area of the rooted subtree seeded in a link (the sum of the areas of the connected links) is the cumulative drainage area at the link; a stream is a collection of links defined by Strahler's ordering procedure [7].]

All the above scaling characteristics as well as other geomorphological "laws" were consistently obtained as consequences of the joint application of three principles of optimal energy expenditure [2]: (*1*) minimum energy expenditure in any link of the network, (*2*) equal energy expenditure per unit area of channel anywhere in the network, and (*3*) minimum energy expenditure in the network as a whole. The application of the first two principles gives that the rate of energy expenditure E_i in the ith link of length L_i transporting a flow rate Q_i may be expressed as $E_i \propto L_i Q_i^{0.5}$. The third principle implies that the topological arrangement of the links of a drainage network will be such as to minimize $\sum_i L_i Q_i^{0.5}$. Using the total drainage area to a link, A_i, as a surrogate of the mean annual flow Q_i [2], the characteristics of OCN's are obtained from minimization of the total energy expenditure $E \propto \sum_i A_i^{0.5}$ (L_i being the unit in unbiased lattice models [2]). The optimization strategy implemented is similar to random search techniques developed for the traveling salesman problem [2].

One example is shown in Fig. 1 where starting from a random initial condition the optimization converges to an

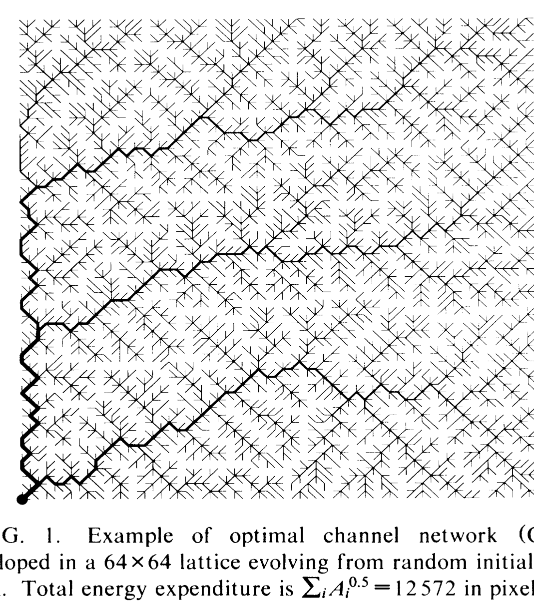

FIG. 1. Example of optimal channel network (OCN) developed in a 64×64 lattice evolving from random initial planform. Total energy expenditure is $\sum_i A_i^{0.5} = 12\,572$ in pixel units [2]. The irregular large-scale arrangement is reminiscent of the random initial condition. Many minima of energy expenditure are known to exist [2] whose configurations differ in large-scale structure and nevertheless yield matching statistics.

OCN whose statistics match extremely well all experimental features. In all cases tested, as in Fig. 1, the optimized probability distributions of stream length and total cumulative area always evolve into power laws with matching slope for the range of logarithmic scales allowed by the lattice size, independently of the initial conditions and of the shape of the region in which minimization of E takes place. OCN results are also robust with respect to perturbation and quenched randomness [2]. This resilience is strongly reminiscent of self-organized criticality [3].

A synthesis of the agreement of demanding statistics of experimental data and OCN's obtained from very different geometries and initial characterizations is provided by the geomorphological width functions $W(x)$. $W(x)$ is defined as the probability measure obtained by dividing the number of links at given distance x from the outlet by the total number of links in the network (x being measured along the network and normalized by the maximum path distance along the streams from source to outlet). $W(x)$ is directly linked with the runoff response of a basin [6]. It is produced by the multiplicative process of partitioning the total drainage area by the spatial organization of the network along the 1D support of the longest path. The relative proportion of area in Δx scales as $\propto \Delta x^{\alpha}$ and the number of sites where the same proportion occurs scales as $\propto \Delta x^{f(\alpha)}$. Figure 2 shows the match of multifractal spectra $f(\alpha)$ vs α [1,8] of the width function of real channel networks from DEM's and of OCN's [2].

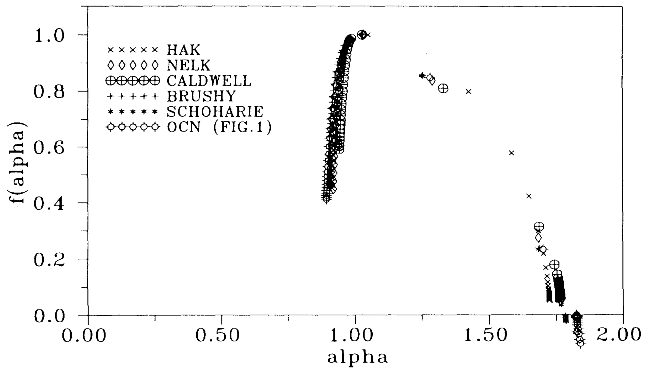

FIG. 2. Multifractal spectra $f(\alpha)$ vs α of the width function $W(x)$ of real river networks from DEM's (Hak, 1260 km^2; Caldwell, 993 km^2; Nelk, 440 km^2; Brushy, 322 km^2; Schoharie, 2408 km^2), and of the OCN of Fig. 1.

A reasonable model of erosion in cohesive soils is given by the dynamic evolution $\partial z/\partial t = f(\tau - \tau_c)$, with $f(x) = 0$ for $x \leq 0$ (where τ is the local shear stress and τ_c is a critical threshold for erosion). The shear stress produced by the flow rate Q in dynamic equilibrium is $\tau = \rho g y \nabla z$, where ρ is the density of water, g the acceleration of gravity, ∇z the local slope, and y the flow depth (approximating the ratio of flow area and wetted perimeter) which scales, because of optimality as in (*1*) and (*2*) [2], as $y \propto Q^{0.5}$. Thus at the ith site $\tau_i \propto Q_i^{0.5} \nabla z_i$. Whenever shear stresses exceed τ_c anywhere in the network we expect activity [9]. This identifies an interesting problem because the transcendence of the critical threshold is dependent on the self-organizing structure itself since the local flow rate $Q_i \approx A_i$ depends on the patterns linked by the developing network. Recent models of self-organized criticality [10,11] recognize that the transfer of interactions to the neighbors of a site where the threshold value is exceeded may depend on the state of the system.

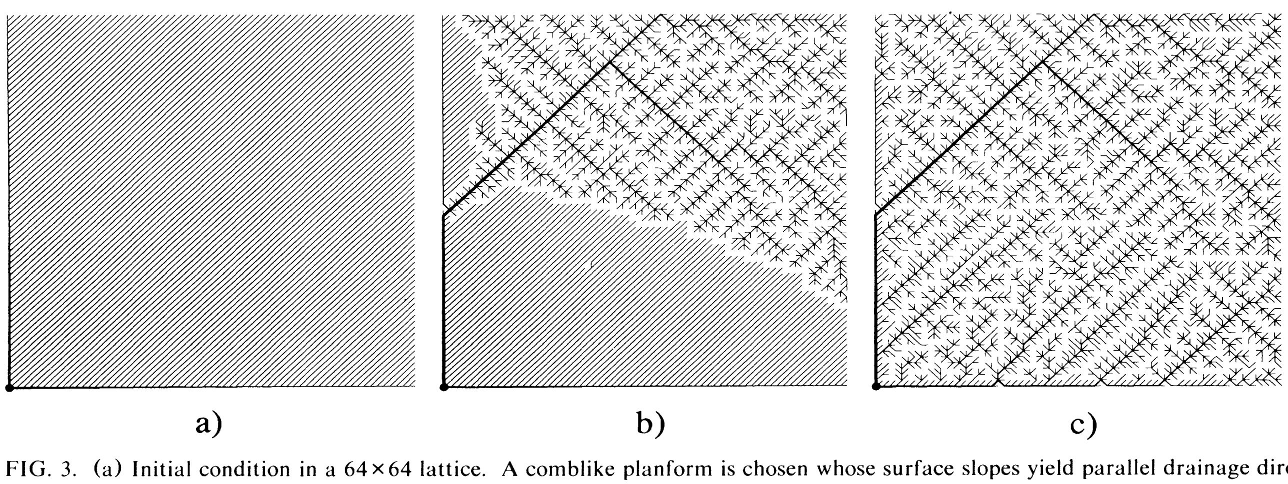

FIG. 3. (a) Initial condition in a 64×64 lattice. A comblike planform is chosen whose surface slopes yield parallel drainage directions and, at the left and bottom boundaries, two collecting channels leading to the common outlet at the bottom left corner. The slopes are everywhere critical ($\Delta z_i = \tau_c / A_i^{0.5}$ with $\tau_c = 1$ in pixel units [2]). $\sum_i A_i^{0.5} = 20845$. (b) Intermediate configuration after some perturbations and the related readjustments ($\sum_i A_i^{0.5} = 13930$). (c) Stable stationary structure after 10^3 perturbations failed to modify the network ($\sum_i A_i^{0.5} = 12579$). The relatively regular large-scale arrangement of the channel structure resembles the regularity of the initial condition and yields statistics identical of those of Fig. 1, including bifurcation and length ratios [7], drainage density, and Melton's and Moon's law [2,6].

We now suggest the equivalence of structural stability and the enforcement of the global principle of minimum energy dissipation by defining the following model of self-organizing structure. We choose a two-dimensional lattice. Each site i has two variables, z_i, and total flow Q_i, here surrogated by the draining area A_i. For a given initial configuration, time evolution is performed according to the following rules.

(1) Drainage directions are fixed by steepest descent. A given threshold value τ_c is assigned (to simplify matters, small enough to affect all the network) and lattice effects are avoided [2] by assigning unit length to all directions.

(2) The initial set of elevations is chosen indifferently at random or in a systematic way. Random initial conditions may be generated by Eden growth [2] or by sampling elevations removing singularities [10]. For each site i the shear stress is computed as $\tau_i \propto A_i^{0.5} \Delta z_i$, Δz being the drop along the drainage direction (and hence the slope, lengths being assumed as unity).

(3) The threshold transcendences possibly scattered throughout the network are computed. The maximum transcendence is isolated, say at site j, and the elevation of the jth site is reduced to the value which yields $\tau = \tau_c$. The mass released by the lowering of z_j (the avalanche in [3]) is removed from the system because at equilibrium the threshold once exceeded in a site will always be exceeded downstream because of the increasing discharge (other transfer options have also been tested).

(4) Drainage directions are recomputed because they are altered due to the modified elevation of site j. Changes may be minor or extensive, depending on the actual configuration.

(5) Steps (3) and (4) are repeated until no transcendences are isolated. Thus at any stage the studied system evolves to a critical state.

(6) The critical state (5) is perturbed at random by adding elevation to a node. The local flow convergence induced by the perturbation may yield violation of criticality as in (3) and a readjustment of the structure. The system is perturbed at random until further perturbations do not induce variations in the configuration of the system.

Many experiments were performed evolving from random or systematic initial conditions and the system always evolved into a final state characterized by common features. To describe such features, we choose to show (Fig. 3) the evolution of an initial comblike structure, a demanding test imposed by a systematic arrangement whose elevation field z_i yields parallel drainage directions almost everywhere. The elevations are scaling as $\Delta z_1 \propto A_i^{0.5}$ and hence are at the critical threshold everywhere. Thus it is a stationary critical state and the system, except for perturbations, would not evolve; this is far from the behavior expected in nature, in particular because no fractal scalings are shown [Figs. 4(a) and 4(b)].

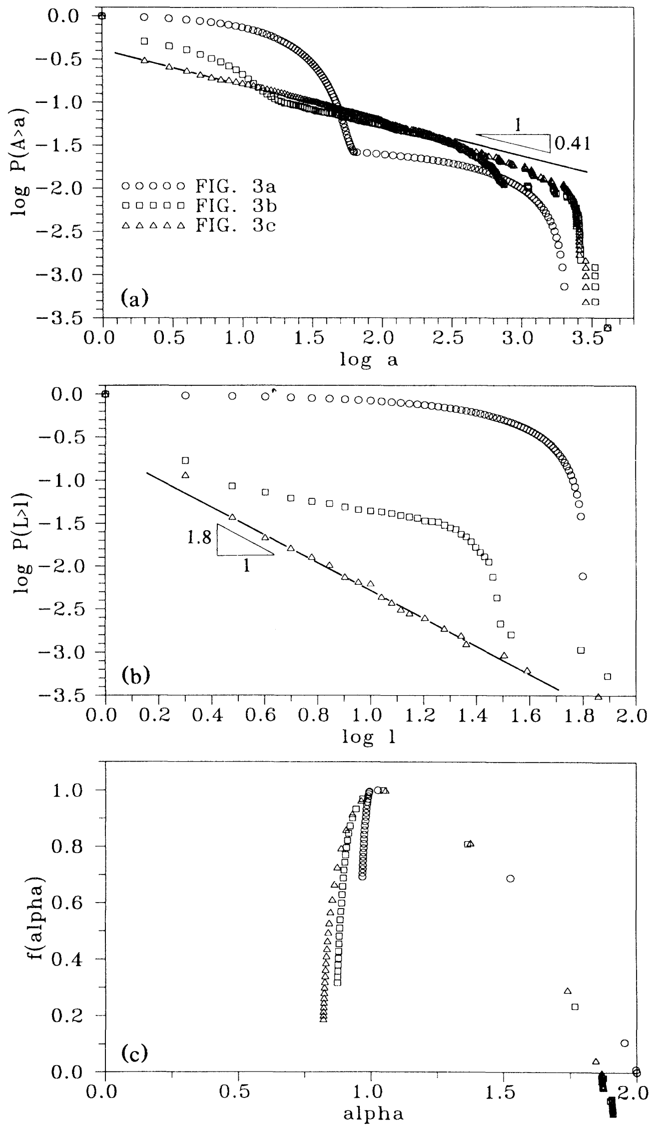

FIG. 4. Statistics of Fig. 3: (a) Cumulative distributions of drainage areas on a log-log scale. We note the nonfractal statistics of the initial structure, straightening progressively to a power law with exponent -0.41 over a wide range of scales. (b) Cumulative distributions of stream lengths on a log-log scale. As in (a) power laws are clearly emerging with exponent -1.8. (c) The evolution of the multifractal spectra of the width functions of the networks. Since the initial structure collapses in a singular $(\alpha, f(\alpha))$ spectrum concentrated in (2,1), the early spectrum corresponds to the structure developed after a few perturbations. The final shape is indistinguishable from that in Fig. 2.

The stable structure after many perturbations reaches a stationary state [Figs. 3(c) and 4] which shows power laws in the probability of transcendences of area and length over all scales allowed by the lattice size. The fractal dimensions (Fig. 4) are close to those found in nature [4,5] and in OCN's [2]. The evolution of the multifractal spectrum [Fig. 4(c)] is also significant. In fact, the network develops in such a manner that the multifractal spectrum of the width function evolves towards a spectrum nearly identical to those of Fig. 2. We also note that the total energy dissipation of the system, $\sum_i A_i^{0.5}$, decreases for every readjustment following a perturbation, and stabilizes to a value very close to that of the OCN in Fig. 1 and of all our 64×64 OCN's. Thus the effect of perturbations on a model of critical threshold results in networks of lower energy characterized by fractal and multifractal scalings whose similarity to those in nature and in OCN's increases progressively.

The explanation for the equivalence of the global criterion of least energy dissipation and the structural stability in a model of self-organized criticality relies on a proof [12] of equivalence of minimizing total rates of energy expenditure and the total potential energy of the system constrained to obey the scaling in elevations postulated by the principles *(1)* and *(2)* which are the ones controlling the functional form of the critical parameter. Let $\sum_i z_i$ (i spans all sites) be a measure of total potential energy of the network. $\sum_i z_i = \sum_i \sum_{j\in\gamma_i} \Delta z_j = \sum_i A_i \Delta z_i$ (where γ_i indexes the sites on the path from the outlet to the ith site) because if the area ruler is unity the cumulative area A_i measures the number of sites connected to the ith link and therefore the number of times every Δz_i appears in the sum. If we constrain potential energy ($\Delta z_i \propto A_i^{-0.5}$) we obtain $\sum_i z_i \propto \sum_i A_i^{0.5}$.

The above results suggest that random perturbations (of tectonic or hydrologic origin) induce erosion activity through which the system readjusts automatically tending irreversibly to lower its mean elevation towards stable configurations. Randomness, either in the initial condition or induced by perturbations, is crucial to the development of fractal scalings. Since the stable structure corresponds to minimum total energy dissipation in the system, OCN's are viewed as models of spatial self-organized criticality.

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