# Soliton Solutions for Free-Electron-Laser Applications

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(Received 19 May 1992)

A new class of nonlinear traveling-wave solutions of the relativistic cold-fluid model are presented for a system consisting of a relativistic cold electron beam propagating through a helical wiggler magnetic field. The solutions, which are in the form of isolated soliton pulses of coupled electromagnetic and plasma waves, are obtained numerically. They represent possible nonlinear saturated states of the free-electron-laser instability and may also have useful applications in particle acceleration schemes.

PACS numbers: 41.60.Cr, 52.35.Sb, 52.40.Db, 52.75.Di

The free-electron laser (FEL), which generates coherent electromagnetic radiation using an intense relativistic electron beam propagating through a static transverse magnetic field, has been the subject of several theoretical and experimental investigations [1]. While many simple model calculations exist to analytically estimate the linear growth rate of the instability, very few calculations, other than those dependent on large computer simulations and numerical codes, have addressed the nonlinear development and saturation of the instability. In particular, there are no simple model descriptions of the nonlinear saturated state. A notable exception is the work of Lane and Davidson [2] where large-amplitude traveling-wave solutions were obtained for the nonlinear Vlasov-Maxwell equations. These BGK (Bernstein-Greene-Kruskal) type solutions are stationary in a frame of reference moving with the ponderomotive phase velocity. An interesting question that arises is whether there are simpler fluid analogs of these BGK type solutions, especially finite pulse solutions like solitons. To the best of our knowledge such solutions have not been found for the one-dimensional relativistic cold-fluid model of the FEL, although the model has been extensively employed for linear calculations. This is somewhat surprising since traveling-wave solutions (including soliton solutions) of the cold-plasma model have been widely investigated in the context of the nonlinear theory of intense laser-plasma interactions [3–6]. A possible reason for this lacuna in the FEL problem could be the complexity introduced in the fluid model by the presence of the beam and the wiggler field which make a traveling-wave analysis somewhat difficult. In this Letter we carry out such an analysis and find that soliton solutions do indeed exist for the one-dimensional cold-fluid model of the FEL. Our findings are based on numerical solutions of the relativistic fluid equations modeling the nonlinear evolution of perturbations to a relativistic cold electron beam propagating through a helical magnetic wiggler. The solutions are in the form of isolated soliton pulses which occur with a discrete velocity spectrum. This new class of traveling-wave solutions, consisting of intense light pulses coupled to large-amplitude plasma waves, represent possible saturated states of the FEL. They might also form interesting candidates for particle accelerator schemes like the inverse free-electron laser [7] in which the large electrostatic potential of the plasma wave may be used to get significant acceleration of electrons.

We begin with a brief discussion of the relativistic cold-fluid model that has been previously developed by Davidson, Johnston, and Sen [8] for describing the nonlinear evolution of the free-electron-laser instability. For a relativistic cold electron beam of uniform density propagating in the $z$ direction through a helical wiggler magnetic field $\mathbf{B}^{(0)}(\mathbf{x}) = -\hat{B}[\cos(k_0 z)\hat{\mathbf{e}}_x + \sin(k_0 z)\hat{\mathbf{e}}_y]$, the cold-fluid equations consist of the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial (n V_z)}{\partial z} = 0 \tag{1}$$

and the $z$ component of the equation of motion

$$\frac{\partial p_z}{\partial t} = e\frac{\partial \phi}{\partial z} - mc^2 \frac{\partial \gamma}{\partial z}, \tag{2}$$

where $\gamma = (1 + p^2/m^2c^2)^{1/2}$ is the relativistic factor, and $p_z$ is the longitudinal momentum. Note that for one-dimensional variations (in $z$) the symmetry of the problem permits elimination of the transverse momentum by the exact integration $\mathbf{p}_\perp = e\mathbf{A}/c$, where $\mathbf{A}$ is the total vector potential. The electrostatic potential $\phi$ can be eliminated from (2) by a further differentiation in time and making use of Poisson's equation and the $z$ component of Ampere's law, to get

$$\frac{\partial^2 p_z}{\partial t^2} = -mc^2 \frac{\partial \gamma}{\partial t\, \partial z} - \omega_{p0}^2 \left[ \frac{n}{n_0}\frac{p_z}{\gamma} - \frac{p_{0z}}{\gamma_0} \right]. \tag{3}$$

The evolution of the transverse electromagnetic perturbations are given by the inhomogeneous wave equation

$$\left[ \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right] \delta\mathbf{A} = \frac{\omega_{p0}^2}{c^2} \left[ \frac{n\mathbf{A}}{n_0 \gamma} - \frac{\mathbf{A}^{(0)}}{\gamma_0} \right]. \tag{4}$$

Here $\gamma_0$, $\mathbf{A}_0$, $p_0$, and $\omega_{p0}^2 = 4\pi e^2 n_0/m$ denote unperturbed values due to the wiggler magnetic field and the beam. In (3) and (4) the nonlinear terms consist of the striction ($n/n_0$) and relativistic ($\gamma$) nonlinearities. The latter arises from the increase of mass of the electron due to its relativistic quiver motion in the high-frequency field. The consequent decrease in the plasma frequency increases the local refraction coefficient leading to increased concentration of the high-frequency field. The strictional nonlinearity arises from the ponderomotive force which also causes a redistribution of the plasma density. In this case, however, the plasma concentration increases in the region of high field intensity thereby decreasing the local refraction index. Thus in the cold-plasma model these two nonlinearities oppose each other [3,4] and their competition determines whether solutions of the soliton type are possible. For a soliton to form, the favorable relativistic nonlinearity has to overcome the unfavorable tendencies of the striction effect and push electrons out of the high-field region. The pressure of this high-field region is then balanced by the electrostatic restoring force arising from the local expulsion of the electrons. To investigate such solutions it is convenient first to transform Eq. (4) to the so-called wiggler coordinates defined by

$$\begin{aligned}\hat{e}_1 &= \cos(k_0 z)\hat{e}_x + \sin(k_0 z)\hat{e}_y ,\\ \hat{e}_2 &= -\sin(k_0 z)\hat{e}_x + \cos(k_0 z)\hat{e}_y ,\\ \hat{e}_3 &= \hat{e}_z .\end{aligned} \tag{5}$$

The components of Eq. (4) in the new coordinate system take the form

$$\begin{aligned}&\left(\frac{\partial^2}{\partial z^2} - k_0^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta A_1 - 2k_0\frac{\partial \delta A_2}{\partial z} = \frac{\omega_{p0}^2}{c^2}\left(\frac{nA_1}{n_0\gamma} - \frac{A_1^{(0)}}{\gamma_0}\right),\\ &\left(\frac{\partial^2}{\partial z^2} - k_0^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta A_2 - 2k_0\frac{\partial \delta A_1}{\partial z} = \frac{\omega_{p0}^2}{c^2}\frac{nA_2}{n_0\gamma} .\end{aligned} \tag{6}$$

We now make the traveling-wave ansatz that all the dynamical quantities depend on $z$ and $t$ through the combination $\xi = z - ut$, where $u = \text{const}$ is the signal speed. In the wave frame the continuity equation can be readily integrated to give

$$\frac{n}{n_0} = \frac{(\beta - \beta_b)\gamma}{(\gamma\beta - p_3)} , \tag{7}$$

where $\beta = u/c$, $\beta_b = V_0/c$, and $p_3 = p_z/mc$. Using (7) and some straightforward algebra it is now possible to reduce (3),(6) to a set of three coupled nonlinear differential equations for the variables $\rho_1$, $\rho_2$, and $Z = \beta\rho_3 - \gamma$:

$$(1-\beta^2)\ddot{\rho}_1 - 2\dot{\rho}_2 - \rho_1\left\{1 + \frac{\hat{\omega}_p^2\gamma_0|\beta - \beta_b|}{[(\beta^2-1)(1+\rho_1^2+\rho_2^2)+Z^2]^{1/2}}\right\} = -\hat{\omega}_c\gamma_0(1+\hat{\omega}_p^2) , \tag{8}$$

$$(1-\beta^2)\ddot{\rho}_2 + 2\dot{\rho}_1 - \rho_2\left\{1 + \frac{\hat{\omega}_p^2\gamma_0|\beta - \beta_b|}{[(\beta^2-1)(1+\rho_1^2+\rho_2^2)+Z^2]^{1/2}}\right\} = 0 , \tag{9}$$

$$\ddot{Z} + \frac{\hat{\omega}_p^2\gamma_0|\beta - \beta_b|Z}{(\beta^2-1)[(\beta^2-1)(1+\rho_1^2+\rho_2^2)+Z^2]^{1/2}} + \frac{\hat{\omega}_p^2\gamma_0}{(\beta^2-1)}(1-\beta\beta_b) = 0 , \tag{10}$$

where the overdot denotes $d/d\zeta$ and the normalizations are $\hat{\omega}_c = \omega_c/ck_0$, $\hat{\omega}_p = \omega_p/ck_0$, $\zeta = k_0\xi$, and $\rho_\alpha = p_\alpha/mc$ with $\alpha = 1,2,3$. Equations (8)–(10) are the final set of coupled equations that we must solve for studying the nonlinear evolution of the FEL. These equations exhibit many similarities to, and important differences from, traveling-wave equations studied in the past for intense laser-plasma interactions [3–6]. The most striking difference is the inhomogeneous term in Eq. (8) for the $\rho_1$ oscillator, which results from the presence of the wiggler field. The $\beta_b$ contribution arises from the beam velocity. These equations admit one exact integral of motion which may be written as

$$Q = \tfrac{1}{2}[(\beta^2-1)(\dot{\rho}_1^2+\dot{\rho}_2^2)+\dot{Z}^2] + W(\rho_1,\rho_2,Z) = \text{const} , \tag{11}$$

where

$$W(\rho_1,\rho_2,Z) = \tfrac{1}{2}(\rho_1^2+\rho_2^2) + \frac{\hat{\omega}_p^2\gamma_0|\beta-\beta_b|}{\beta^2-1}[(\beta^2-1)(1+\rho_1^2+\rho_2^2)+Z^2]^{1/2} + \frac{\hat{\omega}_p^2\gamma_0(1-\beta\beta_b)}{\beta^2-1}Z - \hat{\omega}_c\gamma_0\rho_1(1+\hat{\omega}_p^2) . \tag{12}$$

The equilibrium solutions ($d/d\zeta = 0$) of (8)–(10) are given by $\rho_{01} = \hat{\omega}_c\gamma_0$, $\rho_2 = 0$, and $Z_0 = \gamma_0(\beta\beta_b - 1)$. A small-signal analysis around the equilibrium yields a traveling-wave dispersion relation [8] which can be formally related to the familiar cold-fluid normal-mode dispersion relation for a helical wiggler FEL. As shown in Davidson, Johnston, and Sen [8], the FEL instability corresponds to the condition $\beta < \beta_b < 1$. We note from (11) that in this case the problem is similar to a Hamiltonian of coupled anharmonic oscillators with three degrees of freedom ($\rho_1,\rho_2,Z$) where the effective mass for two of the anharmonic oscillators is negative. In general there can be a rich variety of solutions including

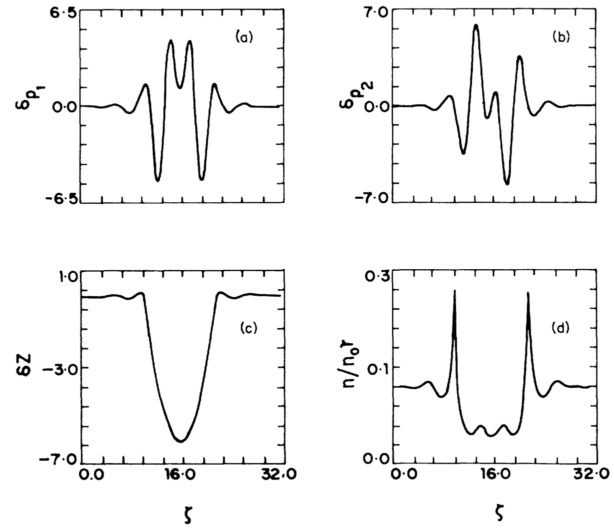


FIG. 1. Profiles of (a) $\delta\rho_1$, (b) $\delta\rho_2$, (c) $\delta Z$, and (d) $n/n_0\gamma$ for a soliton pulse with $\beta=0.5$, $\gamma=10$, $\hat{\omega}_c=0.5$, and $\hat{\omega}_p=0.501935$.

infinite wave trains, solitons, and even aperiodic chaotic solutions [9].

Our primary interest is to investigate soliton solutions for which we have solved Eqs. (8)-(10) numerically and looked for solutions that decay exponentially as $\zeta \to \pm\infty$. We find that such solutions do indeed exist and occur for discrete values of $\hat{\omega}_p$ and $\beta$. In other words, for a fixed value of $\beta$ (and $\hat{\omega}_c$ and $\gamma_0$) finding soliton solutions turns out to be an eigenvalue problem in $\hat{\omega}_p$. The sizes and shapes of these solitons also vary as a function of $\hat{\omega}_p$. Figure 1 shows a typical soliton solution for $\beta=0.5$, $\hat{\omega}_c=0.5$, $\gamma_0=10$, and $\hat{\omega}_p=0.501935$. Typically the $\delta Z=(Z-Z_0)$ profile [Fig. 1(c)] has a characteristic bell shape, whereas $\delta\rho_1=(\rho_1-\rho_{01})$ [Fig. 1(a)] and $\delta\rho_2$ [Fig. 1(b)] have a number of nodes. The $n/n_0\gamma$ profile [Fig. 1(d)] shows that the density is reduced in most of the soliton region but piles up at the edges. This suggests that physically these solitons correspond to trapped light bubble solutions in the beam fluid, in which standing light waves are set up in an "effectively empty" cavity. We see further evidence of this as larger-amplitude solitons appear with increasing $\hat{\omega}_p$ (keeping other quantities fixed). These solitons have a larger spatial extent in $\zeta$ and accommodate larger units of the standing-wave pattern in them. Table I lists some of the eigenvalues (of $\hat{\omega}_p$) at which solitons appear for $\beta=0.5$ and their maximum amplitudes in $\delta Z$. Similar discrete spectra of solitons can be obtained at other values of $\beta$ as well (as long as $\beta<\beta_b$). These solutions therefore constitute a new class of nonlinear stationary solutions for the FEL which can be viewed as fluid analogs of the BGK type solutions. Unlike the kinetic solutions where particle trapping effects play the dominant role, the saturation mechanism here is quite different and results from the nonlinearities due to

TABLE I. Characteristic plasma frequency ($\hat{\omega}_p$) and soliton amplitude ($\delta Z_{\max}$) for $\beta=0.5$, $\hat{\omega}_c=0.5$, and $\gamma=10$.

| | | | | |
|---|---|---|---|---|
| $\hat{\omega}_p$ | 0.5019 | 0.5118 | 0.5138 | 0.5151 |
| $\delta Z_{\max}$ | −6.079 | −18.74 | −31.25 | −71.98 |

striction and relativity effects. When their combined effect is favorable, in the sense of increasing the local refraction index, they can cause a trapping of the electromagnetic wave in a plasma wave excited in the beam due to bunching of the beam density. The pressure of the trapped electromagnetic wave is supported by the space charge electrostatic field of the plasma wave. Since the electrostatic field plays an important role in the formation of the soliton, these solutions can only occur in the Raman regime where collective effects are significant. The typical time scale for the development of these soliton solutions is related to the characteristic evolution times of the nonlinearities involved in the process. Both the relativistic and striction nonlinearities develop on the collective electron response time scale [5], $\omega_p^{-1}$, and the typical growth rate $\Gamma$ for the FEL instability in the Raman regime is approximately $\omega_c(\hat{\omega}_p/\gamma)^{1/2}$ whose value is a fraction of $\omega_p$ and approaches $\omega_p$ at high wiggler amplitudes [1]. The time for an initial potential perturbation $\delta\phi_i$ to grow to an amplitude $\delta\phi_f$ can thus be approximated as $t_s \approx (1/\Gamma)\ln(\delta\phi_f/\delta\phi_i)$. From Poisson's equation, $\delta\phi_i \sim 4\pi e L_s^2 \delta n_i$, where $L_s$ is the spatial dimension of the soliton and $\delta n_i/n$ can be approximately taken as $\approx 1/(nL_s^3)^{1/2}$ for random noise level initial fluctuations. For solitons we need $e\delta\phi_f/mc^2 \sim 1$. Using these conditions and for typical values of the soliton solutions we get $t_s$ to be of the order of a few plasma periods. For solitons to have enough time to form in the device, $t_s$ should be less than $L/c\beta_b$ (the transit time of the electrons through the amplifier-oscillator device of length $L$), i.e., $Lk_0 > \beta_b \hat{\omega}_p^{-1}$. Since $\beta_b \sim 1$ and $Lk_0 \gg 1$, this condition can be easily met in the Raman regime.

To conclude, we have obtained soliton solutions for the first time in the context of the free-electron laser. These one-dimensional nonlinear traveling-wave solutions have been obtained by a numerical solution of the relativistic cold-fluid model equations and are in the form of isolated pulses of coupled intense electromagnetic and plasma waves. We suggest that this new class of solutions can have a large number of applications. They could provide simple model descriptions of the nonlinear saturated state of the free-electron instability in the Raman regime. The possibility of obtaining large electrostatic fields also makes them interesting from the point of view of particle and photon accelerators. The inverse free-electron laser is one such scheme where the large electric fields of the soliton solutions could in principle be used to transfer energy from a laser to a relativistic electron beam in the presence of the magnetic field of an undulator.

It is a pleasure to thank Professor P. K. Kaw for many stimulating discussions during the course of this work. One of us (A.S.) is grateful to Professor Abdus Salam, the IAEA, UNESCO, and SAREC for hospitality at the ICTP, Trieste, where a part of this work was carried out.

[1] T. C. Marshall, *Free Electron Lasers* (MacMillan, New York, 1985).

[2] B. Lane and R. C. Davidson, Phys. Rev. A **27**, 2008 (1983).

[3] A. I. Akhiezer and R. V. Polovin, Zh. Eksp. Teor. Fiz. **30**, 915 (1956) [Sov. Phys. JETP **3**, 696 (1956)].

[4] V. A. Kozlov, A. G. Litvak, and E. V. Suvorov, Zh. Eksp. Teor. Fiz. **76**, 148 (1979) [Sov. Phys. JETP **49**, 75 (1979)].

[5] P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. Lett. **64**, 2011 (1990).

[6] S. C. Wilks, J. M. Dawson, W. B. Mori, T. Katsouleas, and M. E. Jones, Phys. Rev. Lett. **62**, 2600 (1989).

[7] C. Pellegrini, in *The Challenge of Ultrahigh Energies,* Proceedings of the European Committee of Future Accelerators, Rutherford Appleton Laboratory Meeting, Oxford, 1982, edited by J. Mulvey (Rutherford Appleton Laboratory, Didcot, United Kingdom, 1982).

[8] R. C. Davidson, G. L. Johnston, and A. Sen, Phys. Rev. A **34**, 392 (1986).

[9] P. K. Kaw, A. Sen, and E. Valeo, Physica (Amsterdam) **9D**, 96 (1983).