Non-Gaussian Distribution in Random Advection Dynamics

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A discrete model of random transport is analyzed both numerically and theoretically. Non-Gaussian fluctuations are always realized when a finite portion is transported at a time. Gaussian and power-law fluctuations appear in extreme limits.

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In the study of statistical physics for nonequilibrium systems, the deviation from the Gaussian distribution has been the central issue. We expect the existence of universal mechanisms for producing non-Gaussian distributions; however, our knowledge is still at the elementary level.

A typical far-from-Gaussian distribution may be a power law. As a result of the scale invariance or fractality much attention has been paid recently to power-law distributions. Power-law distributions are known to appear at the critical point of a second-order phase transition or at a self-organized critical state [1]. Irreversible particle aggregation processes also tend to produce power-law distributions, as rigorously known in the aggregation system with injection [2]. A mathematical theorem of stable distributions tells us that power-law distributions can also be realized by simple summation of independent random variables with divergent variances [3].

Non-Gaussian distributions are especially important in fluid turbulence [4]. The latest technique of direct observation of turbulence [5] and also numerical integration of the Navier-Stokes equation [6] are clarifying the details of the non-Gaussian nature of the fluctuation of the velocity field. A remarkable result is that the distribution of velocity difference, $\delta\mathbf{v}\equiv\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{x}+\mathbf{r})$, for example, shows tails which are much larger than those of Gaussian distributions but still much smaller than power laws. Scalar quantities in turbulence such as temperature are also known to follow non-Gaussian, non-power-law distributions [7]. Namely, a typical distribution in turbulence is between Gaussian and a power law.

In this paper we introduce a simple model of random transport which can be regarded, in a limit, as a model of passive scalar transport by random advection. We show that the fluctuation of the scalar quantity is generally not Gaussian, and the tail of its distribution is between Gaussian and a power law. Gaussian and power-law distributions appear only in extreme situations.

We consider scalar variables on a discrete space-time, $\omega(r,t)$. A portion of $\omega(r,t)$ is transported randomly to another site at a time described by the following stochastic equation:

$$\omega(r_0,t+\Delta t)=\omega(r_0,t)-j(r_0,t)\omega(r_0,t)+\sum_r S(r_0,r,t)j(r,t)\omega(r,t)\,, \qquad (1)$$

where $j(r,t)$ is a random number in the range $0\le j(r,t)\le 1$ and $S(r_0,r,t)$ is also a random number which takes the value of either 1 or 0. $S(r_0,r,t)=1$ means that there exists transport from r to r_0 at time step t. In order to conserve the total quantity $\Omega=\sum_r\omega(r,t)$, we require that $S(r_0,r,t)$ equals 1 for only one r_0 for each r. Here, we allow the possibility of $S(r,r,t)=1$, which means no transport.

In the case that transport is restricted to neighbors and $j(r,t)$ is close to 0, the variation of $\omega(r,t)$ is expected to be smooth and Eq. (1) is approximated by the diffusion equation

$$\frac{\partial}{\partial t}\omega(r,t)=D\Delta\omega(r,t)\,, \qquad (2)$$

in the continuum limit. Here, the diffusion coefficient D is proportional to $j(r,t)/\Delta t$ (Δt is the time step).

In the other extreme case when $j(r,t)=1$, Eq. (1) describes irreversible aggregation phenomena; for example, in the case where $\omega(r-1,t)$ moves to the rth site keeping all other sites unchanged, we have $\omega(r,t+\Delta t)$

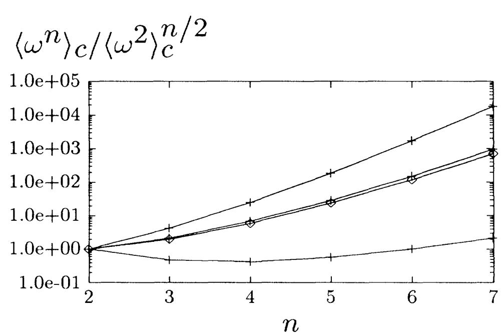

FIG. 1. Exact mean-field values of higher-order cumulants $\langle\omega^n\rangle_c$ normalized by the variance. + for the mean-field model; j=0.1, 0.7, and 0.9, from bottom to top. The exponential distribution is shown for comparison (◇).

$=\omega(r,t)+\omega(r-1,t)$ and $\omega(r-1,t+\Delta t)=0$. Namely, for $0<j(r,t)<1$ Eq. (1) generally represents both diffusion and aggregation effects. In the following analysis, to avoid unessential complexity, we assume that all $j(r,t)$ take a constant value j $(0<j<1)$, and observe how the system changes for different values of j.

We start our study with a mean-field version of Eq. (1) to estimate the effect of the finite-portion transport theoretically. We consider the situation that a site is interacting with a mean-field site and transports to or from the mean-field site occur with probability $\frac{1}{2}$, independently. In this situation the stochastic Eq. (1) becomes

$$\omega(t+t_0)=\begin{cases}\omega(t), & \text{Prob } \frac{1}{4},\\ (1-j)\omega(t), & \text{Prob } \frac{1}{4},\\ \omega(t)+j\omega_M, & \text{Prob } \frac{1}{4},\\ (1-j)\omega(t)+j\omega_M, & \text{Prob } \frac{1}{4},\end{cases} \tag{3}$$

where ω_M is the value at the mean-field site which is defined as an independent random number having the same distribution as $\omega(t)$. By introducing the characteristic function

$$Y(\rho,t)=\int_{-\infty}^{\infty} e^{i\rho\omega}P(\omega,t)d\omega ,$$

where $P(\omega,t)$ is the probability density for ω at time t, Eq. (3) becomes

$$Y(\rho,t+\Delta t)=\tfrac{1}{4}\{Y(\rho,t)+Y(\rho-\rho j,t)\}\{1+Y(\rho j,t)\} . \tag{4}$$

Taylor expansion of Eq. (4) in terms of ρ gives a set of equations for the moment functions $\{\langle\omega^n\rangle\}$, and it can be easily shown that a nontrivial steady state exists when $\langle\omega\rangle\neq 0$, i.e., $\Omega\neq 0$. In the case $\langle\omega\rangle=0$ we will show later that the only steady-state solution is $P(\omega)=\delta(\omega)$; that is, no fluctuation remains.

The steady-state solution for $\langle\omega\rangle\neq 0$ is obtained with the aid of algebraic calculation by computer. In Fig. 1 we plot the cumulants for orders up to 7 together with those for a one-sided exponential distribution for comparison. Higher-order cumulants are not zero in any case and show a tendency to diverge as the order goes to infinity, which clearly demonstrates that the steady-state distribution is not Gaussian.

For smaller j the cumulants are smaller, and in the limit of $j\to 0$ it can be shown theoretically that the normalized cumulant of order n, $\langle\omega^n\rangle_c/\langle\omega^2\rangle_c^{n/2}$, vanishes for $n\geq 3$. Namely, the distribution converges to a Gaussian in this limit, which agrees with the well-known fact that the distribution function satisfying the usual diffusion equation in the continuum limit is a Gaussian. This result indicates that the finiteness of j is very important for the appearance of the non-Gaussian distribution in our random transport dynamics.

In the special case of $j=\frac{1}{2}$ we can estimate the steady-state functional form of $Y(\rho)$ as follows. By denoting $Y_n=Y(\rho 2^n)$, Eq. (4) becomes

$$Y_{n+1}-Y_n=2\frac{Y_n^2-Y_n}{3-Y_n} . \tag{5}$$

Supposing that n can be a continuous number, we approximate the left-hand side by a derivative dY/dn. Then Eq. (5) can easily be integrated and we have the following cubic equation for $Y(\rho)$ with the normalized mean value $\langle\omega\rangle=1$:

$$\rho^2 Y(\rho)^3+[Y(\rho)-1]^2=0 , \tag{6}$$

which can be solved explicitly by Cardano's formula. The asymptotic behavior for $\rho\to\infty$ is given as

$$Y(\rho)\propto\rho^{-2/3} . \tag{7}$$

This power-law decay is very different from the case of a Gaussian distribution where the characteristic function decays faster than any power of ρ. In the case of an exponential distribution the characteristic function is given by $1/(1+i\rho)$ (asymmetric case) or $1/(1+\rho^2)$ (symmetric case), which also shows a power-law decay as $\rho\to\infty$. In this sense the distribution is far from Gaussian, but is similar (but not identical) to the exponential distribution.

Numerical simulations are performed on a one-dimensional lattice with nearest-neighbor transport. A typical system size is 10^4 and the boundary condition is periodic. Figure 2 shows the steady-state cumulative distributions for $j=0.1$, 0.5, and 0.9 in semilogarithmic scale. Here, the initial values for $\{\omega(r,0)\}$ are positive and given randomly. Quite similar results are obtained with other initial conditions, for example, $\omega(r,0)=\text{const}$, so the steady state is expected to be independent of the initial conditions. For $j=0.1$ the tail of the distribution decays quickly like a Gaussian, and for larger j the tail becomes larger as estimated by the mean-field analysis.

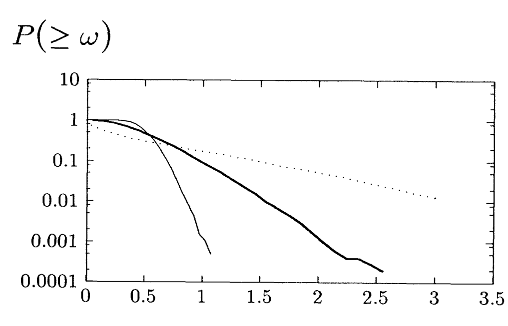

FIG. 2. Semi-logarithmic plot of the steady-state cumulative distributions, $P(\geq\omega)=\int_\omega^\infty P(\omega')d\omega'$, in one dimension with $\langle\omega\rangle>0$. The values of j are 0.3 (thin line), 0.5 (bold line), and 0.9 (dotted line). Averages are taken over ten realizations.

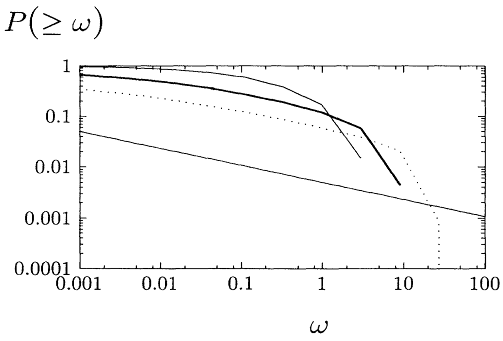

FIG. 3. Log-log plot of the steady-state cumulative distribution in one dimension with $\langle\omega\rangle > 0$. The values of j are 0.9 (thin line), 0.99 (bold line), and 0.999 (dotted line). The slope of the straight line is $-\frac{1}{3}$.

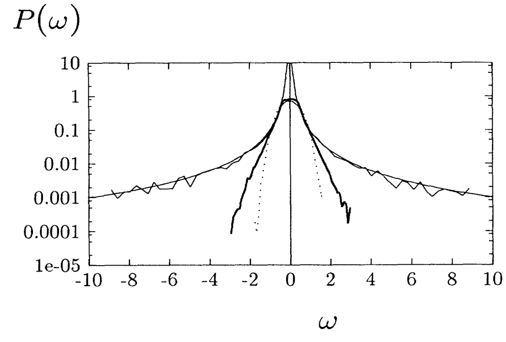

FIG. 4. Semilogarithmic plot of the steady-state probability density $P(\omega)$ in the long-range transport case with $\langle\omega\rangle=0$. Random perturbations in the range $[-1,1]$ are added constantly (the sum of perturbations is controlled to be zero). The parameters are $j=0.3$ (dotted line), $j=0.8$ (bold line), and $j=0.99$ (thin line). The smooth curves show the Lorentzian tails, $P(\omega) \propto \omega^{-2}$.

To see the asymptotic behavior of $j \rightarrow 1$ we plot the cases of $j=0.9$, 0.99, and 0.999 in log-log scale in Fig. 3. The steady-state distribution gradually approaches a power law $P(>\omega) \propto \omega^{-1/3}$. This power-law distribution is known to be the exact solution of the aggregation limit $j=1$ in the presence of positive injection from outside [2]. In our simulation we have no injection; however, in the case that j is close to but not equal to 1, the small portion left behind at every transport, $(1-j)\omega(r,t)$, may play the role of injection while the dominant effect is the aggregation process, and then the system can realize a nearly power-law steady state.

Next, we consider the case of $\langle\omega\rangle=0$, i.e., $\Omega=0$. The second-order term of ρ in the mean-field equation, Eq. (4), gives the following equation for the variance of ω,

$$\langle\omega^2(t+\Delta t)\rangle=(1-j+j^2)\langle\omega^2(t)\rangle . \tag{8}$$

As known immediately from this equation the variance decays exponentially to zero for $0<j<1$, which shows that the only steady state is the trivial state, $P(\omega)=\delta(\omega)$, as we mentioned previously.

The system becomes nontrivial if we apply random external fluctuations. Namely, we add a new term, $I(r,t)$ in Eq. (1) or $I(t)$ in Eq. (3), which is a random variable having zero mean, $\langle I\rangle=0$. Accordingly a new term $\langle I^2\rangle$ is now added to Eq. (8) and the variance converges exponentially to a finite value. The same argument holds for every moment of ω, $\langle\omega^n\rangle$, so we have a nontrivial stable steady distribution $P(\omega)$.

Numerical results are shown in Fig. 4. Here the transport is long ranged, i.e., the probability of $S(r_0,r,t)=1$ is independent of r_0 for each r. For small j the distribution is close to a Gaussian as typically shown in the case of $j=0.3$ in Fig. 4. For larger j the tails become larger as seen in the case of $j=0.8$, and as $j \rightarrow 1$ the distribution converges to Lorentzian tails, $P(\omega) \propto |\omega|^{-2}$ (see the case of $j=0.99$). This result is reasonable since the case of $j=1$ is equivalent to the aggregation system with injection [2].

An unexpected behavior is found in one dimension with nearest-neighbor transport. When no injection is added to the system of $\langle\omega\rangle=0$, the variance converges quickly to 0 as predicted by the mean-field analysis; however, when we add injections of $\langle I\rangle=0$ the variance $\langle\omega^2(t)\rangle$ does not converge to a finite value but shows a tendency to diverge. This strange behavior may be caused by the peculiarity of one dimension, like the segregation phenomenon [8], but the details are yet to be clarified.

In the case that the variance vanishes or diverges, we have no steady fluctuation, but we can observe a snapshot of fluctuations at a finite time. It is confirmed numerically that in any case of $\langle\omega\rangle=0$ the distribution of ω converges if it is normalized by the square root of the variance, $\langle\omega^2(t)\rangle^{1/2}$. The functional form of the converged distribution is very similar to that of Fig. 4; namely, it is close to a Gaussian for small j, the tails become larger for larger j, and for j very close to 1 the tails nearly follow power laws.

Summarizing the results, we introduced a model of random transport and found that non-Gaussian distributions always dominate if a finite portion is transported at a time. A Gaussian distribution is realized only in the limit of infinitesimal transport. In the other limit that nearly the entire portion is transported at a time, the distribution shows a long tail close to a power law which is a typical distribution in aggregation phenomenon. We have analyzed only the mean-field and one-dimensional cases; two- and three-dimensional cases are to be analyzed in the near future.

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