New Random Matrix Theory of Scattering in Mesoscopic Systems

Keith Slevin[1] and Taro Nagao[2]

[1]*Department of Physics, Toho University, Miyama 2-2-1, Funabashi, Chiba 274, Japan*

[2]*Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan*

(Received 5 October 1992)

A previously unnoticed group structure for the transfer matrices of mesoscopic systems is identified. From the invariant measure for this group and a maximum entropy ansatz we derive the Laguerre ensemble of random matrices. A comparison with the Hofstadter model shows that the Laguerre ensemble provides a quantitative description of scattering in mesoscopic systems under applied magnetic fields.

PACS numbers: 72.10.−d, 02.50.−r, 72.15.−v

The electronic and magnetic properties of mesoscopic systems have been the object of considerable theoretical and experimental study in the field of solid-state physics. At low temperatures (typically millikelvin) and small system sizes (typically nm) mesoscopic systems can be modeled as phase coherent elastic scatterers and inelastic and other phase breaking processes, ignored or included by naive averaging arguments. Numerous novel physical phenomena are observed or are predicted in this regime: universal conductance fluctuations (UCF), Aharonov-Bohm conductance oscillations, chaotic boundary scattering, persistent currents, and anomalous magnetic susceptibilities. In attempting to explain the first of these, UCF, Imry [1] made a connection with the theory of random matrices. This was further developed by Mello *et al.* [2] and Muttalib *et al.* [3]. Underlying both approaches was a particular group structure in the scattering theory. In this Letter we identify a second, distinct and as far as we know previously unnoticed group structure in the problem which leads to an alternative random matrix theory of the scattering. From the group structure, and a maximum entropy ansatz, we show that the elastic scattering in a disordered mesoscopic system can be quantitatively described by the Laguerre ensemble of random matrix theory.

The scattering in a mesoscopic system may be described by a $2N\times 2N$ transfer matrix T relating the incoming and outgoing electron fluxes at say the left (i,o) of the system to those at the right (i',o'),

$$T\begin{pmatrix} i \\ o \end{pmatrix} = \begin{pmatrix} o' \\ i' \end{pmatrix},$$

where N refers to the number of propagating channels. We wish to determine the probability distribution $P(T)$ of the transfer matrix (or some related matrix) given a random distribution of elastic scatterers. This is expressible as a product of two factors $P(T)=p(T)\mu(dT)$ where $p(T)$ is a probability density and $\mu(dT)$ is a measure or "volume element." This separates the analysis into two stages. First the choice of the measure and second the determination of the density. If the matrices T form a group G we can fix $\mu(dT)$ uniquely by demanding that it satisfy $\mu(dT)=\mu(dT')$ if $T'=T*T_0$ for any fixed $T_0\in G$ and $\forall T\in G$, i.e., that the measure be invariant under the group operation, denoted by $*$, for all elements of the group. A unique group measure exists for every group and provides a natural choice for $\mu(dT)$ based on considerations of symmetry alone.

In the present problem there are two symmetries to be considered; conservation of flux and time-reversal symmetry. The first, satisfied under very general circumstances, forces the T matrices to be symplectic. More explicitly the T matrices must satisfy

$$I_c = TI_cT^\dagger, \quad I_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The second, time-reversal symmetry, may be broken by the application of a magnetic field. For simplicity we consider first the case where time-reversal symmetry is broken, returning to the time-reversal symmetric case later. It is central to the previous approaches [2,3] that the matrices T satisfying (1) form a group under multiplication. The invariant measure for this group is [2]

$$\mu(dT) = C\prod_{i<j}|\lambda_i-\lambda_j|^2\prod_{i=1}^{N}d\lambda_i\mu(d[u_1])\prod_{i=2}^{4}\mu(du_i), \tag{2}$$

where C is a constant. The parameters $\lambda_1,\ldots,\lambda_N$ occur in the decomposition of T as [4]

$$T = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}\begin{pmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{pmatrix}\begin{pmatrix} u_3 & 0 \\ 0 & u_4 \end{pmatrix}, \tag{3}$$

where the u's are $N\times N$ unitary matrices and λ is an $N\times N$ positive definite diagonal matrix with elements $\lambda_1,\ldots,\lambda_N$. The decomposition is not unique but it can be made so by fixing the phases of the first element in each column of u_1. The factor $\mu(du)$ denotes the invariant measure for the unitary group [5] and $\mu(d[u_1])$ is a related measure, discussed by Bronk [5], which takes account of the N additional constraints on u_1. The physical meaning of the parameters $\lambda_1,\ldots,\lambda_N$ can be made clearer by calculating the conductance g of the system. For two probe measurements we have [6]

$$g = \mathrm{tr}[tt^\dagger] = \sum_{i=1}^{N}\frac{1}{1+\lambda_i}$$

with g in units of e^2/h.

Unfortunately, though the T form a group under multiplication, the density of the variable λ does not correspond to that of any standard random matrix ensemble. Muttalib *et al.* [3] overcame this difficulty by resorting to a very general class of ensemble introduced by Balian [7]. The density $p(T)$ is fixed by supposing that the ensemble average level density $\sigma(\lambda)$ is known *a priori*. The entropy of the ensemble S,

$$S[p(T)] = \int p(T) \ln p(T) \mu(dT) ,$$

is then maximized subject to this constraint. Note that this procedure does not introduce any further correlation terms into $P(t)$ other than those already appearing in $\mu(dT)$. This makes the importance of the group structure clear: The correlations are determined solely by the group structure and its associated invariant measure. While this would not hold for more general constraints, in practice these are not usually tractable.

We are not in a position to explain our new contribution. We have identified a second group structure in the problem and found that this leads to an alternative random matrix theory of the scattering statistics. Using the invariant measure of this group and a maximum entropy ansatz we derive a quantitative description of the scattering in terms of the Laguerre ensemble of random matrix theory. Consider the matrix $\Omega = \ln TT^\dagger$. From (3) and the properties of the singular value decomposition [8] we can show that the space of matrices Ω is identical to that of matrices of the form

$$\Omega = \begin{pmatrix} 0 & \omega \\ \omega^\dagger & 0 \end{pmatrix} , \tag{4}$$

with ω an $N \times N$ arbitrary complex matrix. Since T is symplectic the eigenvalues of Ω occur in pairs of opposite sign and are $(v, -v)$, where the v are related to the λ defined in the decomposition (3) by

$$\lambda = \frac{\cosh v - 1}{2} .$$

Under addition such matrices form a group, the invariant measure of which is

$$\mu(d\Omega) = \prod_{i,j}^{N} d\omega_{i,j}^R d\omega_{i,j}^I ,$$

where R and I stand for real and imaginary parts. This is identical to [5]

$$\mu(d\Omega) = C \prod_{i<j} |v_i^2 - v_j^2|^2 \prod_{i=1}^{N} v_i dv_i \mu(d[u_1]) \mu(du_2) . \tag{5}$$

Since matrix multiplication is not commutative the group formed by Ω under addition is not isomorphic to the group formed by T under multiplication and so the invariant measures (2) and (5) are not equivalent.

Having determined the invariant measure for the matrices Ω we must now determine their probability density $p(\Omega)$. To do so we shall invoke a maximum entropy ansatz. We are not able to give any *a priori* justification for our ansatz but we shall show that it yields an accurate description of the diffusive regime. We impose as a constraint a fixed value of $\langle \frac{1}{2} \operatorname{tr} \Omega^2 \rangle$, where angular brackets refer to an ensemble average, and maximize the entropy of the ensemble $S[p(\Omega)]$ [7]. If the only symmetry constraint on the matrices Ω were that they be Hermitian, this procedure would yield the Gaussian unitary ensemble. For matrices of the form (4) the result is related to another standard ensemble: the Laguerre unitary ensemble (LUE) [5]. We find

$$P(\Omega) = Ce^{-a \operatorname{tr} \Omega^2} \mu(d\Omega) ,$$

which is our central result. Integration over the unitary matrices u gives the distribution of the v,

$$P(v_1, \ldots, v_N) = C \prod_{i>j} |v_i^2 - v_j^2|^\beta \prod_{i=1}^{N} v_i e^{-\beta a v_i^2} \prod_{i=1}^{N} dv_i , \tag{6}$$

with $\beta = 2$. An obvious change of variable transforms this into the usual form of the LUE distribution. The distribution is completely determined by a single parameter a. In principle this is related to the condition on the trace, $\langle \operatorname{tr} \Omega^2 \rangle = N^2/a$, but we can also regard a as being determined by $\langle g \rangle$. In the large N limit $\langle g \rangle \simeq (4/\pi)\sqrt{2Na}$.

We shall now attempt to justify the maximum entropy ansatz by comparing the LUE distribution (6) with the distribution obtained from a microscopic Hamiltonian model: the Hofstadter model [9]. We consider a bar geometry $[1 \le x \le L_x, 1 \le y \le L_y]$ which is infinite in z and calculate the transmission matrix for electrons incident from $z < 0$ to $z > L_z$ using a Green's-function technique [10]. In the region $1 < z < L_z$ the diagonal Hamiltonian matrix elements are taken as random with a uniform distribution of width W about the origin. The Fermi energy is set at the band center and the magnetic field B is measured in units of flux quanta per lattice cell.

The density $\sigma(v)$ for the LUE is given in terms of the Laguerre polynomials

$$\sigma(v) = 4ave^{-av^2} \sum_{n=0}^{N-1} L_n^2[2av^2] .$$

In Fig. 1 the densities of the LUE and the Hofstadter model are compared for a system in the diffusive regime, $\langle g \rangle = 3.4$. The results shown are for a 2D sample; similar results are obtained for 3D and quasi-1D samples. The parameter a is determined from $\langle g \rangle$. Quantitative agreement is obtained especially for the smaller v which dominate the conductance. The disagreement at large v is absent when T can be written as a product of transfer matrices associated with scatterers in layers at each z. For the Hofstadter model this occurs when $N = L_x L_y$, a condition not satisfied in general. In the large N limit the asymptotic form for the level density is a quarter circle on the positive v axis.

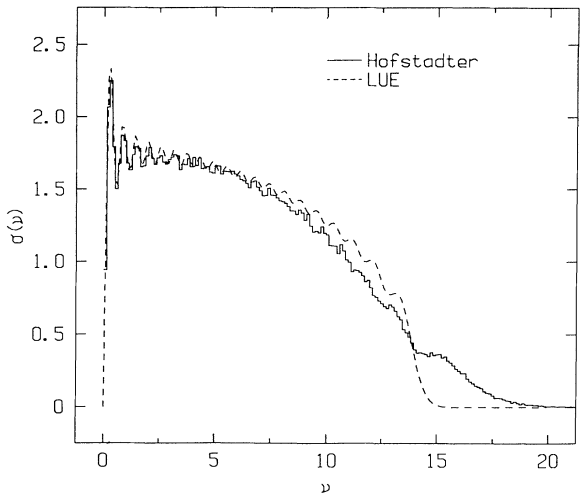

FIG. 1. The density $\sigma(v)$ for the LUE and the Hoftstadter model with parameters $L_x=40$, $L_y=1$, $L_z=40$, $B=0.02$, $W=2$, and $N=20$.

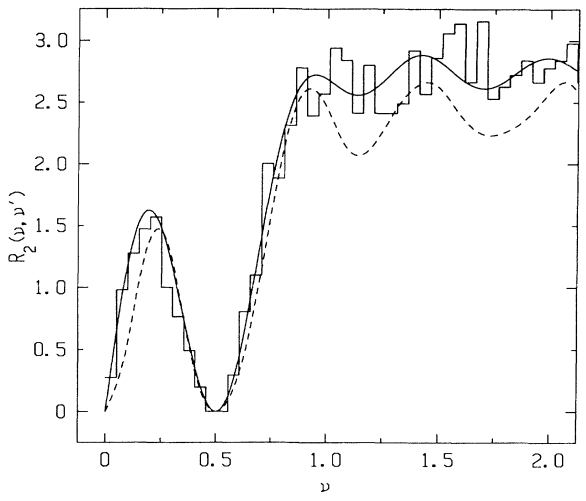

FIG. 2. The two point correlations $R_2(v,v')$ with $v'=0.5$ for the Hofstadter model (histogram), the LUE (solid line), and the theory of Muttalib *et al.* (dashed line). The Hofstadter model parameters are the same as in Fig. 1.

The two point correlation function, $R_2(v,v')$, is obtained from (6) by integration over $N-2$ variables. For the LUE a simple expression exists in terms of the Laguerre polynomials [11]. A typical result is shown in Fig. 2. A quadratic eigenvalue repulsion typical of systems without time-reversal symmetry is evident. For comparison we give the result of the theory of Muttalib *et al.* [3] calculated by taking the density in Fig. 1 as the constraint. The new theory is in slightly better agreement with the numerical data. Note also that the calculation of the correlations with the theory of Muttalib *et al.* requires a prior knowledge of an entire function, $\sigma(\lambda)$. Their calculation with the new theory, making use of a standard ensemble, requires a prior knowledge of only the mean conductance $\langle g\rangle$.

A more stringent test of the quantitative accuracy of the LUE correlation function is the calculation of var(g) which involves the integration of R_2 over the full range of v,v'. For $g\gg 1$ and $N\gg 1$ the result is independent of g so that the LUE reproduces the observed UCF in agreement with Imry's [1] original proposal. The calculated magnitude of the fluctuations is in close agreement with diagrammatic perturbation theory [12] and numerical data for the Hofstadter model in the quasi-1D limit, suggesting that the correlation function may be exact in this limit. The shape dependence of var(g) is not recovered since only a single parameter equivalent to $\langle g\rangle$ appears in the distribution.

In the absence of an applied flux the system will be time reversal invariant and fall into one of two symmetry classes, orthogonal or symplectic. We leave the details for a longer work but the distribution is as (6) with $\beta=1$ for the orthogonal symmetry (LOE) and $\beta=4$ in the symplectic case (LSE). The two cases are distinguished by the strength of spin orbit scattering in the system, weak or absent for the orthogonal case and strong for the symplectic case. The expressions for the density and correlation functions are more complicated but a simple result for the spacing of $v_{\min}=\min(v_1,\ldots,v_N)$ from the origin can be obtained,

$$p(v_{\min})=2N\beta a v_{\min} e^{-N\beta a v_{\min}^2}.$$

In contrast to the spacing between consecutive v, which is given by the Wigner surmise, the form is unchanged by the breaking of time-reversal symmetry. This prediction is checked in Fig. 3 where the distribution of the smallest

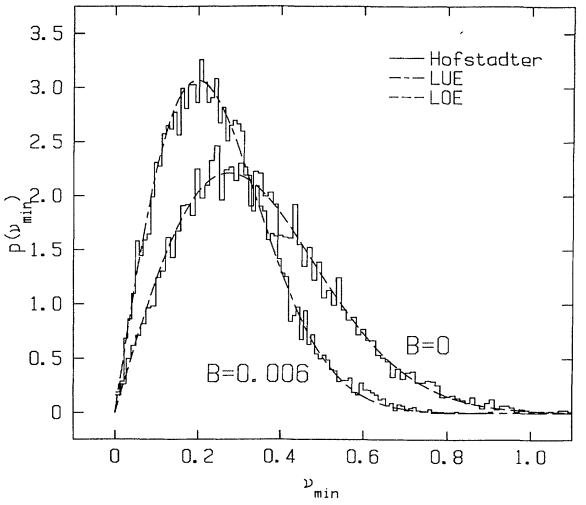

FIG. 3. The distribution of $v_{\min}$ for the Laguerre unitary (LUE) and orthogonal ensembles (LOE) and for the Hofstadter model with parameters $L_x=40$, $L_y=1$, $L_z=40$, $W=2$, $N=40$, and magnetic fields as shown.

v is compared before and after an orthogonal to unitary transition driven by a magnetic field. As predicted, only the scale, not the form, changes.

We discuss briefly the limitations of the LUE. The lack of any prior justification for our choice of constraint means that our discussion must be somewhat empirical. In fact, our choice of constraint was motivated by the observation in the course, of course of computer simulations, that in the diffusive regime the density $\sigma(v)$ can be very well approximated by that of the LUE. It is clear, however, that in other regimes the LUE is not appropriate. First in the long length limit, $L \gg \xi$ where ξ is the localization length the spectrum becomes more rigid than is consistent with the LUE in accordance with Oseledec's theorem [13]. Second on the localized side of the metal insulator transition, the strong disorder limit, a gap appears in $\sigma(v)$ at the origin. There is no such gap in the LUE spectrum. It is possible that the imposition of additional or alternative constraints in the maximization of the entropy would yield the correct distribution in these cases.

To conclude a new random matrix theory of elastic scattering in disordered mesoscopic systems was proposed. The new theory, closely related to the Laguerre ensemble of random matrices, permits a simpler description of the scattering statistics than was possible hitherto.

[1] Y. Imry, Europhys. Lett. **1**, 246 (1986).
[2] P. Mello *et al.*, Ann. Phys. (N.Y.) **181**, 296 (1988).
[3] K. Muttalib *et al.*, Phys. Rev. Lett. **59**, 2475 (1987).
[4] P. Mello *et al.*, J. Phys. (Paris) **1**, 493 (1991).
[5] B. Bronk, J. Math. Phys. **6**, 228 (1965).
[6] D. Fisher *et al.*, Phys. Rev. Lett. **23**, 6851 (1981).
[7] R. Balian, Nuovo Cimento B **57**, 183 (1968).
[8] Any complex matrix ω can be decomposed as $\omega = u_1 v u_2$ where the u's are unitary matrices and v is positive real diagonal matrix [5]. The decomposition can be made unique by fixing the phase of the first element in each column of u_1, a total of N additional constraints.
[9] D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[10] T. Ando, Phys. Rev. B **44**, 8017 (1991).
[11] M. Mehta, *Random Matrices* (Academic, New York, 1991).
[12] P. Lee *et al.*, Phys. Rev. B **35**, 1039 (1987).
[13] V. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1968).