Universal Correlation in the Spectra of Disordered Systems with an Aharonov-Bohm Flux

Aaron Szafer and Boris L. Altshuler
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 April 1991)

We consider the unfolded spectra of disordered systems with the topology of a ring, in the presence of an Aharonov-Bohm flux. We study the flux dependence of a single energy level. We find that the correlator of the derivatives at two different fluxes has a universal functional form, and propose that the result is valid for all systems with Wigner-Dyson statistics.

PACS numbers: 05.45.+b, 71.25.−s, 73.20.Dx

In recent years, there has been great interest in finding criteria to characterize quantum mechanically systems that are classically chaotic [1]. An interesting result that has emerged concerns the statistics of the energy levels of those systems. In the absence of symmetries the level distribution typically resembles that of the eigenvalues of random matrices chosen from Gaussian ensembles [2]. More specifically, the normalized distance between levels, $s = (E_{N+1} - E_N)/\Delta$ (Δ is the mean level spacing at E_N), follows approximately the distribution law $P(s) \sim s^\beta \exp[-\alpha s^2]$, where $\alpha = \pi/4, 4/\pi$ and $\beta = 1, 2$ depending on whether or not the Hamiltonian is T invariant [1, 3], and provided s is smaller than a system-dependent constant $s_c > 1$.

In this Letter we study the spectra of individual quantum particles diffusing in a multidimensional random static potential (disorder). Energy is the only conserved quantity in these systems, therefore they are a subclass of all chaotic systems. We shall propose later that the results we derive for this particular class may remain valid for all systems that follow the Wigner-Dyson statistics $P(s)$.

We consider systems with the topology of a ring, threaded by a constant magnetic flux ϕ through the opening [4]. ϕ affects the quantum spectrum while leaving the classical motion undisturbed. In contrast to $P(s)$ which concerns the level statistics for a fixed ϕ, we study the ϕ dependence of single energy levels.

The Nth level $E_N(\phi)$ satisfies $E_N(\phi) = E_N(\phi + 1)$ (ϕ is measured in units of hc/e). It fluctuates with ϕ (see inset in Fig. 1) with a typical flux-independent amplitude Δ. It is natural to characterize these fluctuations by the correlation

$$C(\phi_-) \equiv \int_{-1/2}^{1/2} d\phi \left\langle \frac{1}{\Delta^2} \frac{dE_N}{d\phi}(\phi) \frac{dE_N}{d\phi}(\phi + \phi_-) \right\rangle , \tag{1}$$

where $\langle \cdots \rangle$ denotes the average over different disorder configurations and over N. Note that $C(\phi_-) = C(-\phi_-)$; therefore it is sufficient to study $C(\phi_-)$ for $0 \le \phi_- \le 1/2$.

$C(\phi_-)$ involves the flux derivative $dE_N/d\phi$ evaluated at two points separated by ϕ_-. When $\phi_- = 0$, $C(\phi_-)$ is positive and its square root measures the typical slope of $E_N(\phi)$. As ϕ_- increases, the slopes at ϕ and $\phi + \phi_-$ eventually become of opposite sign. Let us denote by ϕ_c the value where $C(\phi_-)$ changes sign [i.e., $C(\phi_c) = 0$]. ϕ_c is a typical distance between slope reversals, and so a characteristic flux for the fluctuations of $E_N(\phi)$.

In general, the mean properties of a disordered system depend on two energy scales, Δ and the Thouless energy $E_{cx} = \hbar D/L_x^2$ (here D is the diffusion constant at energy

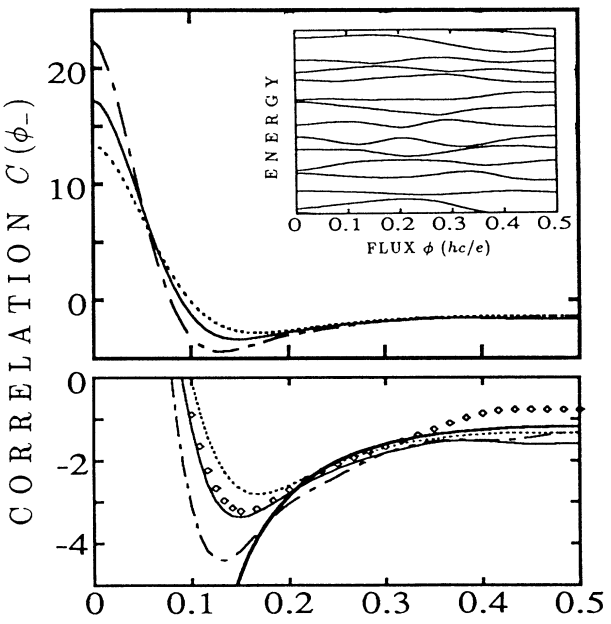

FIG. 1. Top: $C(\phi_-)$ for disordered cylindrical surfaces of size $L_x \times L_y = 27 \times 28$. The random potential has $W = 1.5$ (dashed curve), $W = 1.7$ (solid curve), and $W = 1.9$ (dotted curve). Bottom: Comparison with theoretical curve Eq. (9) (thick solid curve) valid for $\phi_- \gg \phi_c$. Also shown is $C(\phi_-)$ for a single sample with $W = 1.9$ (diamonds). Inset: Energy levels illustrating typical behavior of $E_N(\phi)$.

E_N and L_x characterizes the ring dimension around the flux). For example, its conductance measured in units of $e^2/\hbar$ is $g = E_{cx}/\Delta$ [5]. ϕ_c depends on g, and the functional form $\phi_c(g)$ can be found from the behavior of the levels at small ϕ. Because of T symmetry all levels reverse their slope at $\phi = 0$. To estimate the flux at the next slope reversal, recall that the typical curvature of the levels at $\phi = 0$ is of order E_{cx} [5]. Then $\Delta \sim E_{cx}\phi_-^2$ yields the result $\phi_c \sim \sqrt{\Delta/E_{cx}} \sim 1/\sqrt{g}$.

We study diagramatically the behavior of $C(\phi_-)$ for $\phi_- \gg \phi_c$ (i.e., after several slope reversals). We find a characteristic monotonic loss of correlation with increasing ϕ_-, later shown (numerically) to be valid soon after the first reversal. In principle one can also study the correlation of the energy derivatives for $\phi_- \lesssim \phi_c$ using the supersymmetry technique of Efetov [6]. In this regime $C(\phi_-)$ must depend on g since ϕ_c does. In contrast, we shall prove that when $\phi_c \ll \phi_- \ll 1/2$, $C(\phi_-)$ is given by

$$C(\phi_-) = -\frac{1}{\pi^2}\frac{1}{\phi_-^2} \quad . \tag{2}$$

Remarkably, the right-hand side of Eq. (2) is independent of any property of the system. This means that, provided $\phi_c \ll 1/2$ (i.e., $g \gg 1$), all disordered systems in the diffusive regime satisfy Eq. (2) in some flux interval. Then, as $P(s)$, Eq. (2) *offers a universal quantum mechanical characterization of these systems.*

To derive Eq. (2), consider two energy levels N, M with $N \gg M \gg 1$. Denoting by $n(\phi,\varepsilon)$ the density of states at flux ϕ and energy ε, by definition $N - M = \int_{E_M(\phi)}^{E_N(\phi)} n(\phi,\varepsilon)d\varepsilon = \text{const}$ for all ϕ. Upon differentiation we obtain $(dE_L/d\phi)n(\phi,E_L)|_{L=M}^{L=N} = -\int_{E_M}^{E_N(\phi)}(\partial n/\partial\phi)d\varepsilon$. Consider now an ensemble of diffusive systems with different impurity configurations. Let $\tilde{E}_{M,N}$ and $\tilde{n}(\varepsilon)$ denote the averages of $E_{M,N}$ and $n(\phi,\varepsilon)$ over *both* the disorder ensemble *and* the flux $\phi \in [-1/2, 1/2]$. Writing $E_{M,N} = \tilde{E}_{M,N} + \delta E_{M,N}$ and $n = \tilde{n} + \delta n$, and recalling that in the average $\delta E_{M,N} \ll \tilde{E}_{M,N}$ and $\delta n \ll \tilde{n}$ [7] we arrive at

$$\frac{dE_L}{d\phi}\tilde{n}(\tilde{E}_L)\Big|_{L=M}^{L=N} \simeq -\int_{\tilde{E}_M}^{\tilde{E}_N}\frac{\partial(\delta n)}{\partial\phi}d\varepsilon \quad . \tag{3}$$

Using Eq. (3) and neglecting correlations between off-diagonal terms we express $C(\phi_-)$ in terms of the impurity averaged density-density correlator, $K(\phi,\varepsilon,\phi',\varepsilon') = \langle\delta n(\phi,\varepsilon)\delta n(\phi',\varepsilon')\rangle$, as $C(\phi_-) = \frac{1}{2}\int d\phi \int_{\tilde{E}_M}^{\tilde{E}_N} d\varepsilon\, d\varepsilon'\, \partial^2 K/\partial\phi\,\partial\phi'$ [8]. Our evaluation of this formula follows Ref. [9]. One assumes weak disorder, $E_{M,N}\tau_{el} \gg \hbar$ (τ_{el} is the mean free time), and small energy differences, $\hbar\omega = \varepsilon' - \varepsilon \ll \hbar/\tau_{el}$. The most important contributions to K result from the summation of the usual diffuson and Cooperon diagrams. One obtains $K = K_- + K_+$,

$$K_\pm(\phi_\pm,\varepsilon,\varepsilon') = \frac{1}{2\pi^2\hbar^2}\mathrm{Re}\sum_q (Dq_\pm^2 - i\omega)^{-2} \quad , \tag{4}$$

where $Dq_\pm^2$ are the eigenvalues of diffusion equations involving the flux sum and difference $\phi_\pm = \phi' \pm \phi$. They correspond to eigenfunctions with vanishing normal derivatives at the boundaries of the system [9]. [For a cylindrical surface of dimension $L_x \times L_y$ with periodic boundary conditions in the x direction we have $q_\pm = ((n_x + \phi_\pm)2\pi/L_x, n_y\pi/L_y)$, $n_x = 0, \pm1, \pm2, \ldots$ and $n_y = 0, 1, 2, \ldots$.] The sum in Eq. (4) converges in less than four dimensions. The formalism used to derive Eq. (4) neglects the detailed level structure at scales finer than Δ, so Eq. (4) is valid only when $\hbar Dq_\pm^2 \gg \Delta$ for all $q_\pm$. This results in the condition $\phi_\pm \gg \sqrt{\Delta/E_{cx}} = 1/\sqrt{g} \sim \phi_c$.

Quite generally, K_+ does not contribute to $C(\phi_-)$. Indeed, from $\partial^2 K/\partial\phi\,\partial\phi' = \partial^2 K_+/\partial\phi_+^2 - \partial^2 K_-/\partial\phi_-^2$, and the periodicity of K_+ with ϕ_+, one finds that the flux average of $\partial^2 K/\partial\phi\,\partial\phi'$ equals $-\partial^2 K_-/\partial\phi_-^2$; therefore

$$C(\phi_-) = -\frac{1}{2}\int_{\tilde{E}_M}^{\tilde{E}_N} d\varepsilon\, d\varepsilon' \frac{\partial^2 K_-}{\partial\phi_-^2}(\phi_-,\varepsilon,\varepsilon') \quad . \tag{5}$$

The flux derivative of K_- yields a rapidly convergent series [see Eq. (4)] where only the first few terms are important. Those terms become small when $\hbar\omega \gtrsim E_{cx}$. Since $E_{cx} \ll \hbar/\tau_{el} \ll \tilde{E}_{M,N}$, the main contribution to the integral in Eq. (5) comes from $\omega \ll 1/\tau_{el}$; therefore Eq. (4) can be used. After the energy integrations we find

$$C(\phi_-) = -\frac{1}{4\pi^2}\sum_q \frac{\partial^2}{\partial\phi_-^2}\ln\{1 + [(\tilde{E}_N - \tilde{E}_M)/\hbar Dq_-^2]^2\} \quad . \tag{6}$$

To obtain Eq. (2) we consider small ϕ_-'s. At $\phi_- = 0$ the constant solution is always a solution to the boundary value problem with eigenvalue zero, regardless of the system's geometry. Moreover, it is the ground-state eigenfunction, therefore, it is nondegenerate. (In cylindrical systems it corresponds to $n_x = n_y = 0$.) T symmetry forbids the eigenvalue to change linearly with ϕ_-. Hence, at small ϕ_- Eq. (6) is dominated by a term of the form

$$C(\phi_-) \simeq -\frac{1}{4\pi^2}\frac{\partial^2}{\partial\phi_-^2}\ln[(\tilde{E}_N - \tilde{E}_M)/\alpha\phi_-^2]^2 \quad , \tag{7}$$

where α is a constant. Upon differentiation the system-dependent constants disappear, and the universal form Eq. (2) results. We conclude that Eq. (2) is valid for any diffusive system with $\phi_c \ll 1/2$ regardless of its particular characteristics, even its shape.

For the specific case of a system defined on a cylindrical surface of dimension $L_x \times L_y$, it is easy to find all the eigenvalues of the diffusion equation. In this case, Eq. (6) can be evaluated explicitly to obtain a formula that is valid also when $\phi_- \simeq 1/2$. Rearranging terms and using $\tilde{E}_N - \tilde{E}_M \gg \hbar Dq_-^2$ in Eq. (6) one finds

$$C(\phi_-) = -\frac{1}{\pi}\sum_{n_y=0}^{+\infty}\left[\sum_{n_x=-\infty}^{+\infty}\frac{(n_x+\phi_-)^2-(n_yL_x/2L_y)^2}{[(n_x+\phi_-)^2+(n_yL_x/2L_y)^2]^2}\right] . \quad (8)$$

Carrying out the summation over n_x in Eq. (8) by using contour integration finally yields

$$C(\phi_-) = -2\left[\frac{1}{1-\cos 2\pi\phi_-}+\sum_{n_y=1}^{\infty}\frac{1-\cosh(n_y\pi L_x/L_y)\cos 2\pi\phi_-}{[\cosh(n_y\pi L_x/L_y)-\cos 2\pi\phi_-]^2}\right] . \quad (9)$$

Equation (9) is universal as it does not depend on E_{cx} or Δ. Contrary to Eq. (2), however, it depends on the system's geometry (through L_x/L_y). Of course, in the limit $\phi_- \ll 1/2$ Eq. (9) reduces to Eq. (2).

Having derived our main formula Eq. (2) and its generalization to cylindrical surfaces Eq. (9), we proceed to test the results numerically. We use a tight binding model with homogeneous on site disorder in the range $[-W/2, W/2]$ (the hopping matrix elements have absolute value 1). The calculation involves diagonalizing the Hamiltonian for each given sample at each given flux. To satisfy $\phi_c \ll \phi_- \ll 1/2$ over a reasonably large interval of fluxes ϕ_- requires very large g values and hence very large sample sizes. We therefore test Eq. (9) which is valid for $\phi_c \ll \phi_- \leq 1/2$. Of course, once the applicability of Eq. (9) is established, the assumptions and approximations used in its analytical derivation are justified and the validity of the more restrictive Eq. (2) follows.

Figure 1 shows $C(\phi_-)$ for samples with three different g values. In each case we neglect the "low-g states" at the top and bottom of the band and consider 600 eigenvalues at its center. The ensemble averages are over ten samples. Because of the rapid convergence of the series in Eq. (9) only a few of the lowest modes contribute to $C(\phi_-)$; hence the formula derived in the continuum can be used to fit the results in the lattice. The numerical results confirm the universality of $C(\phi_-)$ for $\phi_- \gg \phi_c$ and are in good agreement with Eq. (9).

The diamonds in Fig. 1 correspond to a single sample with $W = 1.9$. Although ensemble averaging gives better statistics and so enhances the agreement with theory, clearly it is not essential. This is consistent with the ergodic hypothesis [10] according to which disorder and spectral averages are equivalent. When $g \gg 1$ the function $E_N(\phi)$ is expected to have many fluctuations in the interval $-1/2 \leq \phi \leq 1/2$, making the flux averaging in $C(\phi_-)$ very effective. In that case, it is possible that Eq. (2) will be valid for each single level without any further averaging.

Equation (2) was derived for disordered systems. Since it does not contain the mean free path l, it is reasonable to ask whether it remains valid when l reaches the system size. In this case boundary scattering, rather than bulk disorder, is the relevant scattering mechanism. If the boundary has no symmetries, the classical system is expected to be chaotic and, since energy is the only conserved quantum number, the spectrum will show level repulsion as in the case of disordered systems. If the boundary has symmetries, the classical problem will be partially integrable and the quantum spectrum will have level crossings. Is $C(\phi_-)$ sensitive to the different behaviors in those quantum billiards? We test the applicability of Eq. (2) by studying the spectra of systems of various shapes. The average in Eq. (1) is calculated over each spectrum.

As a nonchaotic billiard we consider a cylindrical surface of dimension $L_x \times L_y = 27 \times 28$ and calculate analytically the flux dependence of its first 2500 levels. The inset in Fig. 2 compares $C(\phi_-)$ to the theoretical Eq. (2). The disagreement is evident.

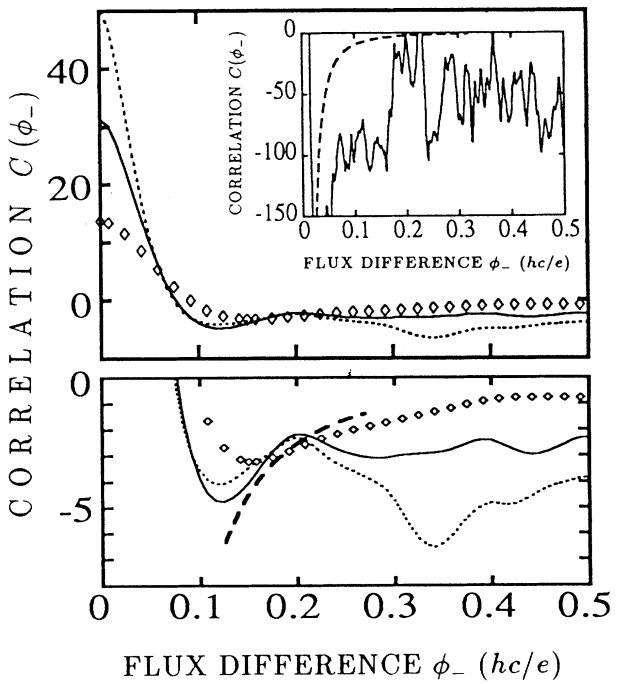

FIG. 2. Top: Typical $C(\phi_-)$ for chaotic billiards. Dotted and solid curves correspond to systems (b) and (c) in Fig. 3, respectively. Diamonds are $C(\phi_-)$ for the single sample with $W = 1.9$ shown in Fig. 1. Bottom: Comparison with theoretical curve Eq. (2) (thick dashed curve) valid for $\phi_c \ll \phi_- \ll 1/2$. Inset: Comparison of Eq. (2) (dashed curve) to $C(\phi_-)$ for a nonchaotic cylindrical surface [Fig. 3(a)]. The disagreement is evident.

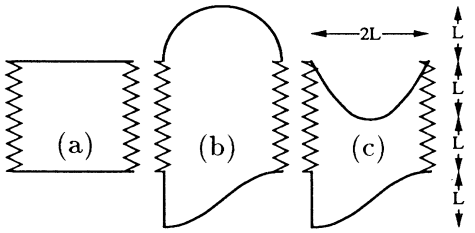

FIG. 3. Some of the billiards considered. The cylindrical surfaces are shown open along the zigzag lines. These lines should be identified. Part (a) is a usual cylinder (nonchaotic). In (b) and (c) bottom edge is negative cosine wave while top edges are convex half circle and concave sine wave, respectively. The dimensions in lattice units for (b) and (c) are $L = 14$ and $L = 18$.

As chaotic billiards we consider several cylindrical surfaces whose upper and lower edges have been modified into simple geometrical curves of different symmetry (see Fig. 3). The spectra are obtained using a tight binding model. To minimize the effects of the lattice, we consider only 250 eigenvalues (~20% of the total) at the bottom of the band, where the wavelengths are typically larger than the lattice unit. We then neglect the first 50 eigenvalues which have "low g." Figure 2 compares typical results for $C(\phi_-)$ to Eq. (2). We find reasonable agreement in the regime of validity of Eq. (2). At larger values of ϕ_- geometry-dependent corrections to Eq. (2) become important and $C(\phi_-)$ becomes system specific. We find systematically that chaotic billiards are close to the theoretical curve while the nonchaotic billiard is not.

In conclusion, the expression of $C(\phi_-)$ derived for disordered systems seems to apply to chaotic billiards as well. This leads us to speculate that Eq. (2) may remain valid for any system whose spectrum at $\phi = 0$ has Wigner-Dyson statistics.

A.S. is grateful to IBM for its generous support, to J. Rhyner and H. Pastawski for their help while learning the physics of disordered metals, and to C. Presilla for helpful discussions. B.L.A. thanks D. Kleppner for interesting conversations, and acknowledges the support of the Joint Services Electronics Program, DAAL 03-89-C-0001, and NSF Grant No. DMR-8719217. We are especially grateful to B. D. Simons for many helpful suggestions. Part of the numerical calculations were done on the IBM 3090 at the Cornell Supercomputing Facility.

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