Stochastic Resonance and Crises

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Stochastic resonance is a phenomenon seen in multistable systems where the addition of noise to the system can amplify small periodic signals. We show in an experiment with a circuit that many of the concepts used to describe crises may also be used to describe stochastic resonance. We speculate that stochastic resonance may be an example of a crisis. This would allow the use of dynamical concepts to describe a process that is usually considered statistically.

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In a typical stochastic resonance situation, a sum of a noise signal and a weak periodic signal is used to drive a bistable system [1,2]. By itself, the noise would cause the bistable system to make transitions randomly between the two states. If the amplitude of the noise signal is near the minimum amplitude required for the system to make transitions, then the probability of making a transition increases nonlinearly as the periodic signal strength increases. If one then filters the resulting signal with a "two-state" filter that only describes which state the system is in, the periodic component of the signal is greatly amplified.

This phenomenon has been shown to be ubiquitous in two-state systems, both in theory and in experiment [1]. The theory describing stochastic resonance is statistical; the noise is usually assumed to be white noise, and the probability of transition is derived from the laws of statistical mechanics [2,3]. The resulting probability depends exponentially on the noise strength.

We wish to view stochastic resonance from a dynamical systems viewpoint. It has been shown that stochastic resonance still occurs when chaos, rather than noise, is used as the nonperiodic component of the driving signal [4]. Chaos is a deterministic dynamical signal. When chaos is used in place of noise for stochastic resonance experiments, the result is not predictable, but it is not random; the entire system is deterministic. This means that the "stochastic" resonance may be described using dynamical, rather than statistical concepts.

We show in an experiment that stochastic resonance may be described in the same way as a crisis. A crisis is a situation in which an attractor collides with the stable manifold of an unstable fixed point [5,6]. This stable manifold is the boundary between two different basins of attraction, so that two separate attractors become one. This description allows one to consider stochastic resonance as a particular case of a much larger group of dynamical phenomena. There is other work describing stochastic resonance near a crisis, but the process itself is still considered statistically in that work [7]. There is also work by Arrechi and co-workers [8,9] describing power spectra generated by noise-induced hopping between different attractors in multistable systems. Our work is similar in many ways to work on noise-induced crises [10] and work on quasiperiodically driven nonlinear systems [11], where some signal added to the driving signal induces a crisis.

Period doubling.— The experiment is based on a periodically driven circuit simulating the Duffing equations, described previously [12]. The circuit is driven so that its response is period doubled, giving it two possible phases, one shifted by one drive cycle from the other. It is possible to make the system shift from one phase to the other by adding noise or chaos to the drive signal. As we have shown previously, this phase flip occurs when the perturbed period-doubled circuit finds itself on an unstable period-one orbit [12]. The period-one orbit persists for one or more cycles, allowing the phase flip to occur. Because we are looking at two different phases as our two states, their symmetry is not affected by inaccuracies in the construction of the circuit.

A signal from a chaotic circuit or noise from a noise generator may be added to the periodic signal driving the Duffing circuit. A second function generator supplies a sinusoidal modulation signal which is also added to the drive. The periodic drive signal generator also provides a sync signal that is used to strobe a digitizer in order to collect a time series consisting of the output of the circuit at a constant phase of the periodic drive. This time series is used to determine the phase of the response. After 4096 cycles, the power spectrum of the phase time series is calculated. This is repeated for twenty time series and the resulting power spectra are averaged together.

To compare the effect of our deterministic chaos to that of noise, we used a chaotic signal from a hysteretic oscillator circuit [13] and white noise from a noise generator. Figure 1 shows the amplitude distributions of both signals. The chaos distribution is irregular, while the noise distribution is approximately Gaussian.

Crises.— The experiment was first done with no modulation signal. The amplitude of the chaos added to the drive signal was first set to a low value, so that the period-doubled system did not flip phase. The amplitude of the chaos was then increased so that flipping did occur, and the average number of cycles between flips was recorded. The average cycles per flip versus chaos ampli-

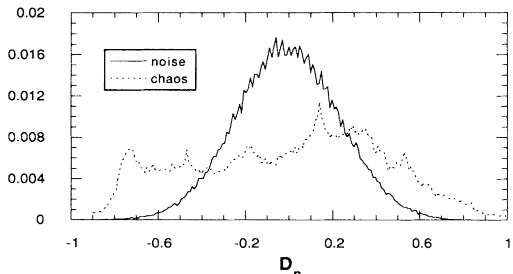

FIG. 1. Amplitude distributions for chaos and noise used to drive the period-doubled Duffing circuit. The normalized amplitude of the signal is D_n and $p(D_n)$ is the probability of the signal having that amplitude.

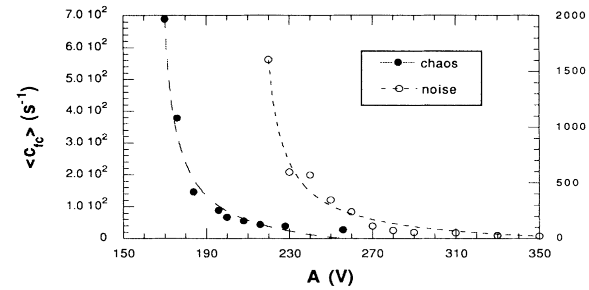

FIG. 2. Average number of drive cycles between phase flips (c_f) vs the amplitude A of the signal added to the periodic drive. The scale with c_{fc} is for chaos, while c_{fn} is for noise. The dashed lines are fits by the critical power law in the text.

tude is plotted in Fig. 2. This situation resembles crisis-induced intermittency [6]. The average number of cycles/flip when white noise is used is also shown in Fig. 2.

Grebogi *et al.* [6] determined that the scaling of the number of transitions between different parts of the attractor after a crisis followed a power law of the form $K/(A-A_c)^\gamma$, where A is our experiment is the noise or chaos amplitude, A_c is the amplitude at which the crisis occurs, and γ is the exponent. It has been shown in experiments that this law also holds for noise-induced crises [10]. If we fit a power law of the form $K_1+K_2/(A-A_c)^\gamma$ to the data in Fig. 2, we find that K_1 is -93, K_2 is 1967, A_c is 166 mV, and γ is 0.68 for chaos and $K_1=-119$, $K_2=8211$, $A_c=213$ mV, and $\gamma=0.83$ for noise. The additive constant K_1 is used to allow for the fact that our algorithm for finding phase flips may report up to twenty spurious flips due to initialization at the beginning of each time series.

Following Grebogi *et al.* [6] the critical exponent γ is determined by the eigenvalues of the unstable period-one orbit whose stable manifold forms the boundary between the basins of attraction of the stable orbits. The exponent will also depend on whether the crisis is homoclinic or heteroclinic. Although we cannot prove which type of crisis exists, we see only the unstable period-one orbit during the transition, so it seems reasonable to assume a homoclinic tangency.

We attempt to apply this theory to our experiment because both the periodic drive signal and the chaos come from deterministic dynamical systems, so the driven Duffing circuit is still a deterministic dynamical system. While it is not the same dynamical system as the periodically driven Duffing, we have shown before that if the added chaos is not too large and the system is stable to the new driving, the dynamics are not greatly changed, so the periodically driven system may be used as an approximation [12]. The factor by which the chaotic signal is multiplied before being added to the periodic signal is a parameter of the system. The theory of critical exponents for crises is not parameter specific; rather it says that given a dynamical system, the critical exponent depends only on the orbits involved in the crisis and not on which parameter is being changed. As long as the dynamical system fits the approximations in Ref. [6], namely, that the tangency is approximately quadratic, then one may calculate the critical exponent for this crisis from the eigenvalues of the unstable period-one orbit.

For a crisis caused by a homoclinic tangency in a two-dimensional system, the value of γ is given by [6]

$$\gamma=\ln|\beta_2|/\ln|\beta_1\beta_2|^2 , \tag{1}$$

where β_1 and β_2 are the unstable and stable eigenvalues of the orbit involved in the crisis.

It was possible to observe the unstable period-one orbit using recently developed techniques for controlling and tracking unstable periodic orbits [14,15]. The system was started at ten slightly different initial conditions near the unstable period-one orbit when the drive signal was at its maximum. The three components of the initial conditions were recorded by a digitizer, as were the three corresponding values 20 μs (about $\frac{1}{75}$th cycle) later. The method of Eckmann *et al.* [16] was then used with the experimental data to find the eigenvalues for the unstable orbit. The three eigenvalues were -1.14, 0.65, and 0.0002. The dynamics were approximately two dimensional here, so Eq. (1) may be used. The resulting value of γ was 0.72, close to the values measured from the power-law fit of 0.68 for chaos and 0.83 for noise. This fits the claim that the flipping of phases occurs due to a crisis.

The fact that we see a critical power law as we would expect for a deterministic crisis even when noise is used is a consequence of the fact that the noise is driving a deterministic system. The noise distribution itself, as pictured in Fig. 1, is close to Gaussian, so that we would expect the number of phase flips to increase exponentially as the noise amplitude increases. This is not the case because the Duffing circuit is not being driven adiabatically by the noise. Rather, the circuit acts as a bandpass filter; noise frequencies near the center of the pass band will have a much larger effect on the circuit than frequencies far

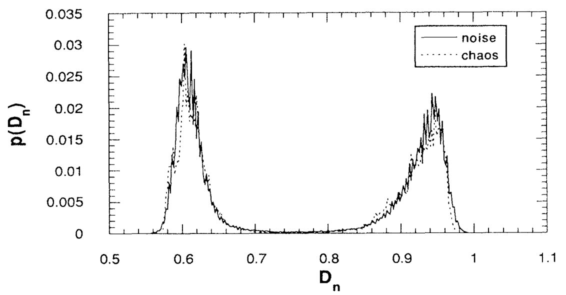

FIG. 3. Probability p of finding the x output from a period-doubled Duffing circuit with the normalized amplitude D_n. The output is strobed with the maximum of the periodic driving signal. White noise or chaos is added to the periodic drive.

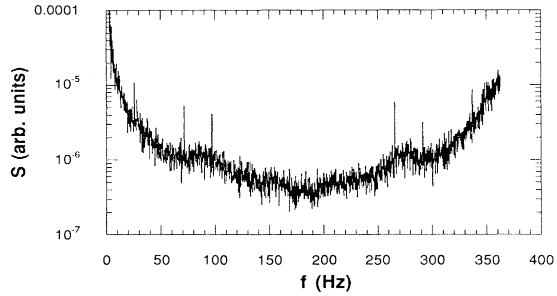

FIG. 4. Power spectrum of the phase time series (S is power) when a chaotic signal and a 100 Hz modulation signal are added to the driving signal for a period-doubled Duffing circuit.

from the center. In Fig. 3 is plotted the actual amplitude distribution of the driven period-doubled Duffing circuit strobed with the maximum of the sinusoidal drive signal. It may be seen that the distributions for the chaos and the noise are almost the same. Least-squares fits confirm that except for an exponential tail for very small probabilities, the distribution for the noisy Duffing follows almost the same power law (with an exponent of approximately 3) as the chaotic driven Duffing circuit. The half width at half maximum of the amplitude distribution for the driven period-doubled circuit also obeys a power law as the noise or chaos amplitude is increased, with an exponent of approximately 1.6.

Stochastic resonance.— We added a small periodic signal (the modulation signal) to the drive. We used a 100 Hz sine wave with an amplitude of 180 mV rms or a 300 Hz sine wave of 72 mV rms. The 100 Hz signal was larger than the 300 Hz signal because the circuit acts as a bandpass filter centered near 700 Hz. The driving signal was about 4.8 V rms.

Figure 4 is a power spectrum of the phase time series for a driving frequency of 100 Hz when the chaos amplitude was 64 mV rms. The spectrum shows peaks at the fundamental frequency and the first odd harmonic, both characteristic of stochastic resonance. Also present are two odd subharmonics at 25 and 75 Hz, and a peak at 265 Hz. This last peak corresponds to the difference between half the driving frequency and the modulation frequency. Because we take our data at the drive frequency, the spectrum only goes up to half the drive frequency. The background spectrum is not Lorentzian, as is typical of other stochastic resonances [2], because the adiabatic approximation does not hold here.

Figure 5 shows the signal-to-noise ratios of the peaks at 100 and 265 Hz as the amplitude of the chaos is increased. The signal-to-noise ratio was found by comparing the amplitude of a peak to the amplitude of the background power spectrum near the peak. The signal-to-noise ratio at 100 Hz drops linearly with the noise, indicating no stochastic resonance, while at 265 Hz, the signal-to-noise ratio increases up to 15 dB (comparable to other experiments [1]) as the chaos amplitude increases, indicating stochastic resonance. The reason that we see stochastic resonance at 265 Hz and not at the fundamental is because we are using a period-doubled system.

It is reasonable to assume that there is some point in the period-doubled orbit where the phase flip is more likely to occur than at any other point. The rate of phase flips for this process should vary like $\sin(f_d/2)$, where f_d is the drive frequency. The probability of flipping is also larger when the modulation signal is at a maximum, so this rate should vary like $\sin(f_m)$, where f_m is the modulation frequency. One rate will modulate the other, so the actual rate is proportional to the product $\sin(f_m)\sin(f_d/2)$, and the resulting frequencies that are amplified are $f_d/2-f_m$ and $f_d/2+f_m$. Because the observed spectrum is limited to frequencies below $f_d/2$, only the lower frequency is seen. We saw similar results when the modulation was 300 Hz and we saw stochastic resonance when noise was used in place of chaos.

In conclusion, the use of chaos allowed us to examine stochastic resonance as a dynamical phenomenon. We found that many of the techniques used to analyze crises in dynamical systems could be used to study stochastic resonance. We also applied the theory of McNamara and Wiesenfeld [2] to our experiments, using a rate law of the form $(A-A_c)^\gamma$ which was related to the crisis

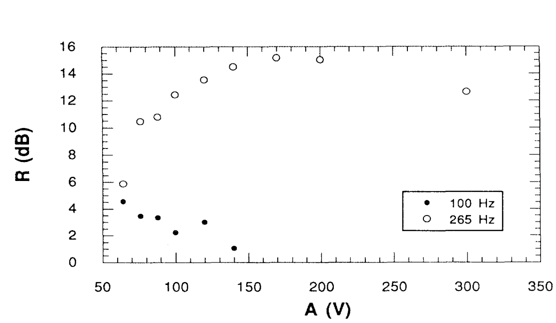

FIG. 5. Signal-to-noise ratio R vs the rms amplitude A of the added chaos for a stochastic resonance experiment in which a 100 Hz modulation signal was added to the noisy periodic drive signal.

power law. The parameters in this rate law were found from fits to the experiments. This type of rate law did produce a stochastic resonance effect. These results will be published later.

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