Compactons: Solitons with Finite Wavelength

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To understand the role of nonlinear dispersion in pattern formation, we introduce and study Korteweg-de Vries-like equations with nonlinear dispersion; $u_t+(u^m)_x+(u^n)_{xxx}=0$, $m,n>1$. The solitary wave solutions of these equations have remarkable properties: They collide elastically, but unlike the Korteweg-de Vries ($m=2$, $n=1$) solitons, they have compact support. When two "compactons" collide, the interaction site is marked by the birth of low-amplitude compacton-anticompacton pairs. These equations seem to have only a finite number of local conservation laws. Nevertheless, the behavior and the stability of these compactons is very similar to that observed in completely integrable systems.

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In this Letter we introduce a class of solitary waves with compact support (which we call compactons) that are solutions of a two-parameter family of fully nonlinear dispersive partial differential equations (PDEs). Compactons are solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the same coherent shape [1,2]. These particlelike waves exhibit elastic collisions that are similar to the soliton interactions associated with completely integrable PDEs supporting an infinite number of conservation laws. However, unlike the soliton collisions in an integrable system, the point where two compactons collide is marked by the creation of low-amplitude compacton-anticompacton pairs.

A prototypical integrable equation with solitons is the quasilinear Korteweg-de Vries (KdV) equation

$$u_t+(u^2)_x+u_{xxx}=0\,.$$

The KdV soliton is proportional to sech^2 and, although it is highly localized in space, it has an infinite span.

Seeking to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, we introduce and study a family of fully nonlinear KdV equations $K(m,n)$:

$$u_t+(u^m)_x+(u^n)_{xxx}=0,\quad m>0,\ 1<n\leq 3\,. \tag{1}$$

These equations, which we denote by $K(m,n)$, have the property that for certain m and n, their solitary wave solutions have compact support. That is, they vanish identically outside a finite core region. In numerical experiments, we have found that these compactons collide as elastically as our numerical experiment is capable of detecting.

Solutions.—The solutions of the $K(2,2)$ equation,

$$u_t+(u^2)_x+(u^2)_{xxx}=0\,, \tag{2a}$$

are typical of the $K(m,n)$ equations and will be described in detail to illustrate this remarkable behavior. [A detailed description of the other $K(m,n)$ equations is in preparation [3].] Traveling-wave solutions $u(\zeta=x-\lambda t)$ for Eq. (2a) satisfy (after two integrations)

$$u_\zeta^2+P(u,P_0)=P_1,\quad P=-\frac{\lambda u}{3}+\frac{u^2}{4}+\frac{P_0}{u^2}\,, \tag{2b}$$

where P_0 and P_1 are constants. Setting P_0 and P_1 to zero leads to a solitary wave with a compact support

$$u_c(x,t)=\frac{4\lambda}{3}\cos^2[(x-\lambda t)/4],\ \text{when}\ |x-\lambda t|\leq 2\pi\,, \tag{3}$$

and $u_c=0$, otherwise.

Although the second derivative of the compacton is discontinuous at its edges, it is a strong solution of Eq. (1) because the third derivative acts on u^2, which has three smooth derivatives everywhere including the edge. The nonlinear dispersion is weaker for small u than the linear dispersion in the KdV equation and allows for the compactification. The compacton's speed depends on its height but, unlike the KdV soliton which narrows as the amplitude (speed) increases, its width is independent of the speed. Because dispersion increases with amplitude, at high amplitudes there is far more dispersion than in the KdV and it can more effectively counterbalance the steepening effect of the nonlinear convection. The invariance of (2a) under $u\rightarrow -u$ and $t\rightarrow -t$ permits negative anticompactons propagating in the opposite direction. Because of their compact structure, neither solitons nor antisolitons interact with each other until the moment of collision.

The quantity D is conserved in Eq. (2a) when we transform it into the form

$$\partial_t D+\partial_x\Phi=0\,.$$

We know of four conservation laws for Eq. (2a) (two of which are quite unusual) and no further *local* conservation laws seem possible [4]:

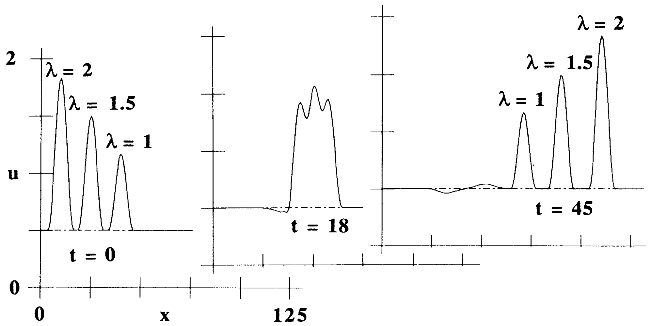

FIG. 1. The evolution of three $K(2,2)$ compactons with speeds $\lambda=2$, 1.5, and 1 starting with centers at $x=10$, 15, and 40.

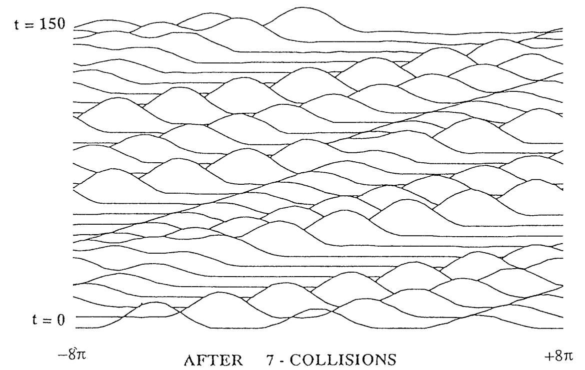

FIG. 2. Space-time plot of three compactons in Fig. 1 on a periodic domain. Note the phase shift whenever they collide.

$$D_1=u,\quad \Phi_1=u^2+(u^2)_{xx}, \tag{4a}$$

$$D_2=u^3,\quad \Phi_2=\tfrac{3}{2}u^4+6u^3u_{xx}, \tag{4b}$$

$$D_{3,4}=u\cos(\beta+x), \tag{4c}$$

$$\Phi_{3,4}=\sin(\beta+x)(u^2)_x+\cos(\beta+x)(u^2)_{xx}. \tag{4d}$$

Though every β is eligible, only two are independent (e.g., $\beta=0$ and $\beta=-\pi/2$). Note also that for compactons

$$\int u_c(x-\lambda t,t)D_{3,4}(x,\beta)dx=0.$$

Figure 1 illustrates the evolution of three compactons with speeds $\lambda=2.0$, 1.5, and 1.0 colliding. The space-time diagram in Fig. 2 shows the phase shift in phase due to the collisions in a similar three-compacton system on a periodic domain. We have performed hundreds of numerical experiments with between two and five interacting compactons and the compactons have always remained intact after the collisions. Even after several dozens of collisions, no radiation is observed, indicating that the collisions are elastic or the radiation is below the numerical accuracy.

In the classical soliton theory, the concepts of integrability and elastic collisions have become synonymous. In our system, even though it is probably not integrable, the interactions are elastic. This suggests that the mechanism responsible for the elastic collisions is probably not integrability.

After the reemergence of compactons in Fig. 1, the collision site is marked by the birth of a small-amplitude, zero-mass, compact ripple which very slowly evolves into compacton-anticompacton pairs. Typically, the maximum amplitudes (and velocities) of newly created compacton-anticompacton pairs are less than 5% of the original compacton's amplitude and therefore they separate on a much longer time scale than the original compacton dynamics. The subsequent pairs are much smaller and take much longer time to form. The number of these pairs is related to the number of interacting compactons. However, we cannot exclude the possibility that new pairs with decreasing amplitudes continue to form indefinitely. In practice, the presence of noise limits the number of pairs which can be observed. In Fig. 3 we show an enlarged picture of the ripple created after the collision of three compactons in Fig. 1. This ripple then decomposes into three clearly observable compacton-anticompacton pairs. When such a ripple is hit by a traveling compacton, the compacton and ripple reemerge unchanged (except for a phase shift) and continue to decompose into compactons.

We also have run several numerical experiments where the initial data were compact but not close to a compacton. These solutions decompose into a number of compactons similar to the example shown in Fig. 4. On the basis of our observations, we tentatively conclude that the compactons for Eq. (2a) play the role of nonlinear local basis functions and that any positive compact initial data can be decomposed into compactons and anticompactons.

There are several numerical difficulties in integrating $K(2,2)$. The nonlinear dispersion prevents us from using the more effective numerical methods commonly used for dispersive systems (such as a spectral split-step method). We used pseudospectral methods in space and a variable order, variable time-step Adams-Bashford-Moulton

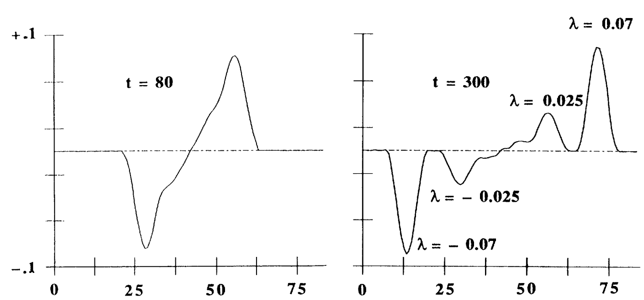

FIG. 3. The evolution of the ripple created by the collision of the three compactons in Fig. 1. Note the slower velocities and the long time it takes for the compactons to emerge.

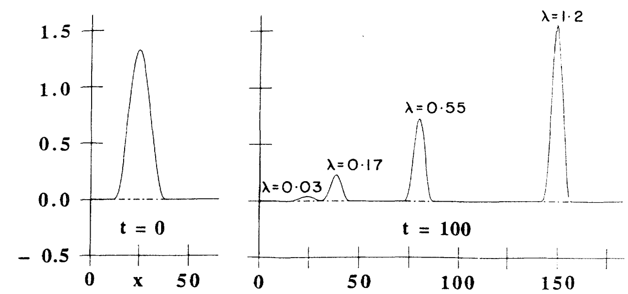

FIG. 4. The initial data $u(x,t=0)=\frac{4}{3}\cos^2(x/2)$ decomposes into four compactons.

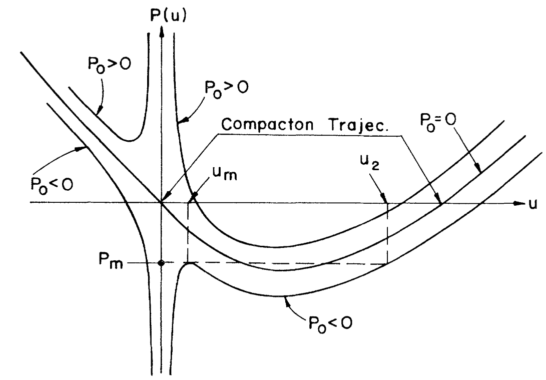

FIG. 5. Phase space of the three distinct classes of traveling waves in the $K(2,2)$ equation; see Eq. (2b). For $P_0>0$, u is either strictly positive or strictly negative. For $P_0<0$, depending on the value of P_1, u is either strictly positive or changes its sign. Note that the local maximum at $u=U_m$ gives rise to shelf solitons. $P_0=0$ is a separating trajectory and describes trigonometric waves which turn into compactons when $P_1=0$.

method in time. The lack of smoothness at the edge of the compacton reduces the spectral method to first order near the edge and introduces dispersive errors into the calculation that are difficult to distinguish from radiation created in a nonelastic collision except by continued mesh refinement. A further numerical difficulty is caused by the delicate balance in the nonlinear dispersion. When expanded, it has a diffusionlike term $2u_x u_{xx}$. On the trailing edge of the compacton $u_x>0$ and this term acts like a destabilizing backward diffusion operator. The solution would be unstable if it were not for the stabilizing nonlinear dispersion. This balance is easily lost in the numerical approximation if the aliasing, due to the nonlinearities, is not handled carefully. Even so, our current numerical schemes fail to calculate the head-on collisions of compactons and anticompactons of similar amplitude. In these collisions, it is not clear whether the instability is due to the numerical aliasing errors, or is caused by a true instability in the $K(2,2)$ equation.

It is clear from Eq. (2b) that there are three distinct classes of traveling periodic waves. The phase space of these waves is shown in Fig. 5. For $P_0 \neq 0$ these waves can be described by elliptic functions. For $P_0=0$ the singular trajectory describes trigonometric waves with the same period of 4π but with an amplitude that varies with P_1. When $P_1=0$, u becomes non-negative and these waves turn into a train of compactons. Because of the degeneracy of $K(2,2)$ at $u=0$, these compactons do not communicate with each other and therefore can be separated.

If $P_0=-P_m$, the potential well on the negative branch of P_0 where $U_m \le u \le U_2$ supports solitary waves around the nonzero state U_m. These "shelf solitons" emerge naturally out of noncompact initial data and seem to collide elastically with each other. Note that as P_0 varies, so does the location of the potential well, yielding a one-parameter family of shelf solitons.

The $K(m,n)$ equations arose in our quest to understand the role of nonlinear dispersion in the formation of nonlinear structures like liquid drops [5]. To derive $K(2,2)$ consider a dense anharmonic chain with many neighbors interaction. The first correction to the continuum leads to an equation of the form

$$u_{tt}=f(u)_{xx}+h^2 g(u)_{xxxx}\,.$$

Here f and g are nonlinear functions that depend upon the details of particular interaction and h is the interparticle distance in equilibrium. In the original work of Zabusky and Kruskal [1], only weakly anharmonic interactions of the nearest neighbors were considered. For small u, f was approximated by a quadratic function and g was approximated by a linear function. If, instead, we assume f and g are monotone functions for small u but at some value U^*, g softens and has a local extremum (e.g., $g=u/[1+u^2]$), then expanding in $v=u-U^*$ results in a Bussinesq equation for v with a quadratic dispersion term in the lowest order. This equation can be expressed as two equations (describing traveling structures to the right or to the left), each of which is a $K(2,2)$ equation. If $f(v)$ is an odd function, the leading-order terms lead to a $K(3,2)$ equation.

Related topics.—The $K(2,2)$ also approximates the second-order difference-differential equation

$$\dot{u}_j+(u_{j+1}+u_{j-1})(u_{j+1}-u_{j-1})/4\Delta x=0\,, \qquad (5)$$

expanded up to $O(\Delta x^2)$. The compact solutions of these discrete equations have only six nonzero values with a trailing shelf behind the discrete compactonlike structure which is very slightly below the value ahead. Other averages than $(u_{j+1}+u_{j-1})/2$ lead to equations, not yet studied, but with compact structures. If the average is replaced by u_j, the resulting equation can be mapped into the Toda lattice but the resulting compactons appear to be unstable. If the discrete part of (5) is modified to $(u_{j+1}^m-u_{j-1}^m)/2\Delta x$, expansion leads to $K(m,m)$ with compactons of the form $\cos^{2/(m-1)}$.

Returning to the general $K(m,n)$ equations, we observe that the solitary waves are compact only if $n > 1$. The singular dispersion at $u=0$ plays a crucial role in the compactification. The upper limit on $n \leq 3$ is necessary for the compactons to be a solution in the classical sense. So far, we have studied four cases $m,n=2,3$ with the compacton solutions ($\zeta = x - \lambda t$),

$$K(2,2):\ u_c = \frac{4\lambda}{3}\cos^2(\zeta/4)\,,$$

$$K(2,3):\ u_c = \text{elliptic function}\,,$$

$$K(3,2):\ u_c = [37.5\lambda - \zeta^2]/30\,,$$

$$K(3,3):\ u_c = \pm\sqrt{3\lambda/2}\cos(\zeta/3)\,.$$

In hundreds of numerical experiments, these compactons also reemerge remarkably the same after colliding with their own kind.

We have found higher dimensional equations that support partially compact structures. For example, the 3D Kadomtsev-Petviashvili equation with nonlinear dispersion,

$$\partial_x[u_t + uu_x + (u^2)_{xxx}] + u_{yy} + u_{zz} = 0\,,$$

also has a compacton solution analogous to (3),

$$u_c = \frac{8\lambda}{3(1+t)}\cos^2\frac{1}{4\sqrt{2}}\left[x + \frac{y^2+z^2}{2(1+t)} - \lambda\ln(1+t)\right],$$

when the argument of cos is $\leq |\pi/2|$, otherwise $u_c = 0$. The support of this solution is an infinite parabolid strip traveling to the right. The solution decays in time and ultimately the support of the solution straightens out into a straight strip.

In summary, we have reported our discovery of solitons with compact support for nonlinear dispersive equations. The robustness of these compactons and the inapplicability of the inverse scattering tools, that worked so well for the KdV, makes it clear that a new mechanism (about which we can say very little at this time) is underlying the processes. Future work will aim at understanding the nonlinear mechanism that causes these structures to be so robust. We have seen that elastic collision is accompanied by the birth of a compact ripple which slowly decomposes into compacton-anticompacton pairs. This event has no counterpart in the conventional soliton theory. Naturally, one would like to find additional applications for compactons, such as in nonlinear optics. Is field theory with particles described by compactons possible?

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